

1 Derivatives in curvilinear coordinates

- (a) Verify that the following function

$$f(x, y, z) = (x^2 - y^2)z$$

satisfies the Laplace equation $\nabla^2 f = 0$ using the expression of the Laplacian in Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

- (b) Convert the function f into spherical coordinates.

Hint: Use the expressions $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, and simplify the expression using the trig identity $\cos^2 t - \sin^2 t = \cos(2t)$.

- (c) Verify that $\nabla^2 f = 0$ using the expression of the Laplacian in spherical coordinates,

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.$$

2 Coordinate basis vectors in curvilinear coordinates vary with position

Consider a particle along the following trajectory in polar coordinates

$$\begin{cases} \rho(t) = e^{kt}, \\ \theta(t) = t. \end{cases}$$

Here k is a positive constant.

- (a) Use the fact that $d\mathbf{r} = d\rho\hat{\boldsymbol{\rho}} + \rho d\theta\hat{\boldsymbol{\theta}}$, find the velocity of the particle and express it in the polar coordinate basis.

- (b) What angle does the velocity make with the $\hat{\boldsymbol{\rho}}$ vector? How does it depend on time?

- (c) What kind of motion does this describe? Sketch this curve.

3 The averaging property of harmonic functions

Suppose f satisfies the Laplace equation, $\nabla^2 f = 0$ in two dimensions, i.e. $(\partial_x^2 + \partial_y^2)f = 0$.

- (a) Use the Taylor expansion to the second order to show that the function f satisfies the following approximate averaging property

$$f(x, y) = \frac{1}{4} [f(x + a, y) + f(x - a, y) + f(x, y + a) + f(x, y - a)] + O(a^3)$$

- (b) *Challenge:* Show that the following *exact* averaging property is satisfied,

$$f(\mathbf{r}_0) = \frac{1}{2\pi R} \oint_C f(\mathbf{r}_0 + \Delta\mathbf{r}) |d\Delta\mathbf{r}|$$

where C is a circle of radius R centered at $\mathbf{r}_0$.

Hint: Consider the integral $\oint_C \frac{g(z)}{z - z_0} dz$ for an analytic function $g(z)$. You may assume that if $f(x, y)$ satisfies the Laplace equation, it is the real part of some complex analytic function $g(z)$.