

1 Line integral

- (a) Consider the vector field $\mathbf{F}(x, y, z) = -x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$. Sketch the projection of this field onto the $z = 0$ plane.

Hint: Try evaluating the field at certain special points first, for example along the coordinate axes and on a grid.

- (b) Consider the curve γ defined by the parametrization $\mathbf{r}(t) = (\cos t) \hat{\mathbf{e}}_x + (\sin t) \hat{\mathbf{e}}_y + t \hat{\mathbf{e}}_z$, for $t \in [0, 2\pi]$. What kind of curve is this? Sketch the curve.

- (c) Evaluate the line integral

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{l}.$$

(d) Is $\mathbf{F}$ conservative? If so, find a potential ϕ for it.

(e) Based on the result of part (d), what is the value of the line integral $\int_{\gamma'} \mathbf{F} \cdot d\mathbf{l}$ where γ' is the line segment from $(1, 0, 0)$ to $(1, 0, 2\pi)$?

2 Parabolic mirror

A parabolic mirror has a reflective surface shaped like a circular paraboloid. When light shines on the parabolic mirror parallel to its axis, all of the incoming light will be reflected towards the focus of the parabola. Thus parabolic reflectors can be used to collect light from distant sources and focus it onto a point.

- (a) Consider the paraboloid given by $x^2 + y^2 - z = 0$, which can be parametrized as $\mathbf{r}(u, v) = (u, v, u^2 + v^2)$. Compute the element of the oriented surface

$$d\boldsymbol{\sigma} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv .$$

- (b) Suppose the energy flux density of an incoming beam of light is represented by the vector field $\mathbf{F} = -\hat{\mathbf{z}}$. What is the flux density of light energy that would be passing through the surface of the mirror at the point $(u, v, u^2 + v^2)$ (if the mirror wasn't reflecting)?

Hint: The flux density through a surface is $\mathbf{F} \cdot \hat{\mathbf{n}}$. So you need to find the normal vector $\hat{\mathbf{n}}$ of the paraboloid surface at $(u, v, u^2 + v^2)$.

- (c) What is the total energy flux focused by the mirror surface below the $z = 1$ plane?

Hint: Integrate the energy density obtained from the previous part over the entire surface below the $z = 1$ plane. What are the bounds on u and v for this integral? What would be a convenient coordinate system for this type of bounds?

- (d) What is the divergence of the energy flux density field $\mathbf{F}$? Using the divergence theorem, find an easier method of computing the total energy flux focused by the mirror.

3 Liquid mirror

One way of obtaining a parabolic reflector described in the previous problem is by spinning a reflective liquid. A liquid rotating at an angular velocity of ω will form into a paraboloid where the height (relative to the lowest point) at radius r is $\omega^2 r^2 / 2g$. In order to create such a mirror with radius R , how much liquid (by volume) do we need?