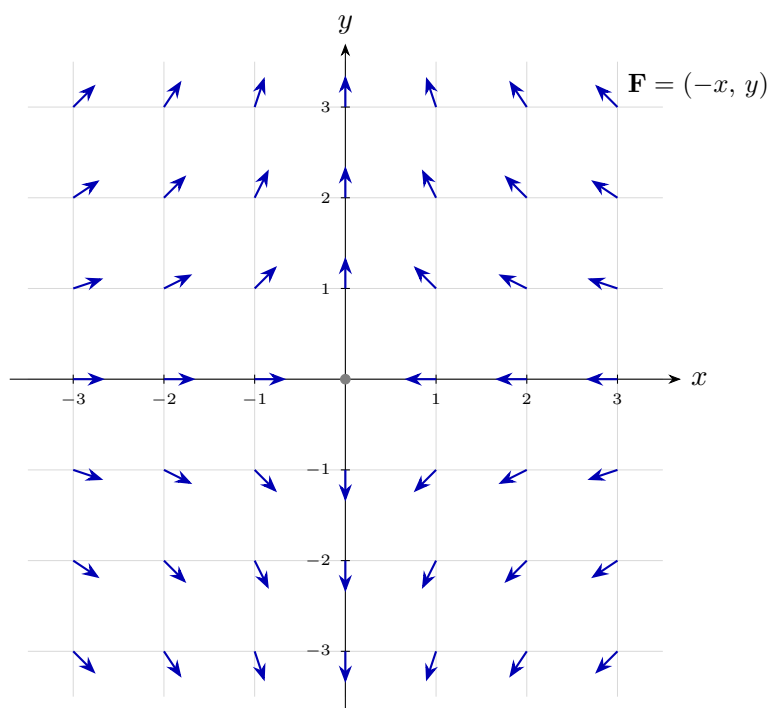


1 Line integral

- (a) Consider the vector field $\mathbf{F}(x, y, z) = -x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$. Sketch the projection of this field onto the $z = 0$ plane.

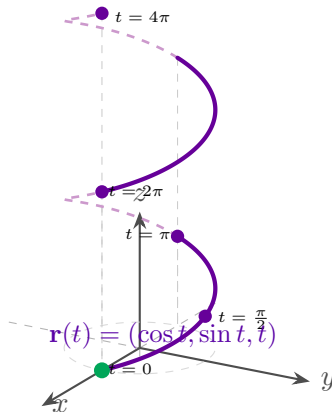
Solution: The sketch of the vector field is shown below:



Note that we need only construct the vector field in the first quadrant. The field on the rest of the plane can be obtained using symmetry. Also note that the flow lines follow the level curves $xy = A$, where A is a constant.

- (b) Consider the curve γ defined by the parametrization $\mathbf{r}(t) = (\cos t) \hat{\mathbf{e}}_x + (\sin t) \hat{\mathbf{e}}_y + t \hat{\mathbf{e}}_z$, for $t \in [0, 2\pi]$. What kind of curve is this? Sketch the curve.

Solution: This is a spiral (helix) of radius 1 and each turn has height 2π .



- (c) Evaluate the line integral

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{l}.$$

Solution: Using the parametrization, we can compute the line element along the curve,

$$d\mathbf{l} = \left(\frac{\partial \cos t}{\partial t}, \frac{\partial \sin t}{\partial t}, \frac{\partial t}{\partial t} \right) dt = (-\sin t, \cos t, 1) dt.$$

Thus $\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{l} = (-\cos t, \sin t, 1) \cdot (-\sin t, \cos t, 1) dt = (2 \sin t \cos t + 1) dt$. Therefore the line integral is equal to

$$\int_0^{2\pi} (2 \sin t \cos t + 1) dt = 2\pi.$$

- (d) Is $\mathbf{F}$ conservative? If so, find a potential ϕ for it.

Solution: Since $\mathbf{F}$ is defined on a simply connected domain and $\nabla \times \mathbf{F} = 0$, it is conservative. A potential function for it is

$$\phi(x, y, z) = \frac{1}{2}(y^2 - x^2) + z.$$

This can be obtained by setting $\phi(0, 0, 0) = 0$ and integrating $\mathbf{F}$ along a path from $(0, 0, 0)$ to (x, y, z) .

Another way to obtain ϕ is to use the fact that $\nabla \phi = \mathbf{F}$, so that $\frac{\partial \phi}{\partial x_i} = F_i$ and solve for ϕ by solving this system of differential equations.

- (e) Based on the result of part (d), what is the value of the line integral $\int_{\gamma'} \mathbf{F} \cdot d\mathbf{l}$ where γ' is the line segment from $(1, 0, 0)$ to $(1, 0, 2\pi)$?

Solution: Since $\mathbf{F}$ is conservative, its line integral does not depend on the path, but only on the end points. The paths γ' and γ have the same end points, so $\int_{\gamma'} \mathbf{F} \cdot d\mathbf{l} = \int_{\gamma} \mathbf{F} \cdot d\mathbf{l} = 2\pi$.

Alternatively, we can use the gradient theorem and evaluate the line integral as

$$\int_{\gamma'} \mathbf{F} \cdot d\mathbf{l} = \phi(1, 0, 2\pi) - \phi(1, 0, 0) = \left(\frac{1}{2}(0 - 1) + 2\pi \right) - \left(\frac{1}{2}(0 - 1) + 0 \right) = 2\pi.$$

2 Parabolic mirror

A parabolic mirror has a reflective surface shaped like a circular paraboloid. When light shines on the parabolic mirror parallel to its axis, all of the incoming light will be reflected towards the focus of the parabola. Thus parabolic reflectors can be used to collect light from distant sources and focus it onto a point.

- (a) Consider the paraboloid given by $x^2 + y^2 - z = 0$, which can be parametrized as $\mathbf{r}(u, v) = (u, v, u^2 + v^2)$. Compute the element of the oriented surface

$$d\boldsymbol{\sigma} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv.$$

Solution: First we evaluate the partial derivatives with respect to the parameters,

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial u}{\partial u}, \frac{\partial v}{\partial u}, \frac{\partial(u^2 + v^2)}{\partial u} \right) = (1, 0, 2u)$$

$$\frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial u}{\partial v}, \frac{\partial v}{\partial v}, \frac{\partial(u^2 + v^2)}{\partial v} \right) = (0, 1, 2v)$$

Therefore,

$$d\boldsymbol{\sigma} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv = (-2u, -2v, 1) du dv.$$

- (b) Suppose the energy flux density of an incoming beam of light is represented by the vector field $\mathbf{F} = -\hat{\mathbf{z}}$. What is the flux density of light energy that would be passing through the surface of the mirror at the point $(u, v, u^2 + v^2)$ (if the mirror wasn't reflecting)?

Solution: To obtain the normal vector $\hat{\mathbf{n}}$ at $(u, v, u^2 + v^2)$, we normalize $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$,

$$\hat{\mathbf{n}} = \frac{(-2u, -2v, 1)}{\sqrt{(2u)^2 + (2v)^2 + 1}} = \frac{(-2u, -2v, 1)}{\sqrt{1 + 4(u^2 + v^2)}}.$$

The flux density is

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{1 + 4(u^2 + v^2)}} (0, 0, -1) \cdot (-2u, -2v, 1) = -\frac{1}{\sqrt{1 + 4(u^2 + v^2)}}.$$

The negative sign shows that the energy is entering the surface, i.e. flowing in the opposite direction to the outward normal vector of the surface.

- (c) What is the total energy flux focused by the mirror surface below the $z = 1$ plane?

Solution: We have computed the element of the oriented surface in part (a). Thus the area element is

$$dA = |d\boldsymbol{\sigma}| = \sqrt{1 + 4(u^2 + v^2)} du dv.$$

Setting $z = 1$ in the parametrization shows us that the $z = 1$ plane intersects the paraboloid at the circle $u^2 + v^2 = 1$. Therefore, the total energy flow is given by the following surface integral

$$\int_D \mathbf{F} \cdot \hat{\mathbf{n}} dA = \int_D \left(-\frac{1}{\sqrt{1 + 4(u^2 + v^2)}} \right) \sqrt{1 + 4(u^2 + v^2)} du dv = - \int_D du dv$$

where D is the unit disc $\{(u, v) \mid u^2 + v^2 \leq 1\}$. Thus the surface integral evaluates to the area of this disc with a minus sign, which is just $-\pi$.

- (d) What is the divergence of the energy flux density field $\mathbf{F}$? Using the divergence theorem, find an easier method of computing the total energy flux focused by the mirror.

Solution: The divergence of $\mathbf{F} = (0, 0, -1)$ is

$$\frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial(-1)}{\partial z} = 0.$$

Thus by the divergence theorem, the flux of $\mathbf{F}$ through any closed surface is zero. Equivalently, the flux of $\mathbf{F}$ through any two surfaces that are bound by the same boundary curve must be equal. This means that the flux of $\mathbf{F}$ through the mirror surface in part (c) is equal to the flux through the disc bounded by the circle $x^2 + y^2 = 1$ on the $z = 1$ plane. Since $\mathbf{F}$ is perpendicular to the $z = 1$ plane, the flux through this disc is just the area of the disc multiplied by the magnitude of $\mathbf{F}$ times -1 , which is $-\pi$. The minus sign accounts for the fact that the direction of $\mathbf{F}$ is opposite to the normal vector of the disc.

3 Liquid mirror

One way of obtaining a parabolic reflector described in the previous problem is by spinning a reflective liquid. A liquid rotating at an angular velocity of ω will form into a paraboloid where the height (relative to the lowest point) at radius r is $\omega^2 r^2 / 2g$. In order to create such a mirror with radius R , how much liquid (by volume) do we need?

Solution: The smallest volume necessary for such a liquid mirror is the volume bounded by the paraboloid, the $z = 0$ plane, and the cylindrical surface $x^2 + y^2 = R^2$. This is given by the integral

$$\iiint_{\substack{x^2+y^2 < R^2 \\ 0 \leq z \leq \omega^2(x^2+y^2)/(2g)}} dx dy dz.$$

Changing the variables x and y into polar coordinates, we can evaluate the above integral as

$$\int_0^{2\pi} d\theta \int_0^R r dr \int_0^{\omega^2 r^2 / 2g} dz = \int_0^{2\pi} d\theta \int_0^R dr \frac{\omega^2 r^3}{2g} = \int_0^{2\pi} d\theta \frac{\omega^2 R^4}{8g} = \frac{\pi \omega^2 R^4}{4g}.$$