

1 Product rules for vector derivatives

Prove the following product rules for vector derivatives. Use Einstein notation to directly compute the results or exploit other known product rules.

(a) $\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f \nabla \cdot \mathbf{A}$

Solution: In Einstein notation $f\mathbf{A}$ is $fA_i\hat{\mathbf{e}}_i$. Taking the dot product with $\nabla = \hat{\mathbf{e}}_i\partial_i$ and expanding using the ordinary product rule, we get

$$\partial_i(fA_i) = (\partial_i f)A_i + f\partial_i A_i.$$

Identifying $\partial_i f$ as the i -th component of ∇f , and $\partial_i A_i$ as $\nabla \cdot \mathbf{A}$, we see that the right hand side is just

$$\nabla f \cdot \mathbf{A} + f \nabla \cdot \mathbf{A}.$$

(b) $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f \nabla \times \mathbf{A}$

Solution: In Einstein notation, $\nabla \times \mathbf{v}$ is $\hat{\mathbf{e}}_i\epsilon_{ijk}\partial_j v_k$ for a vector field $\mathbf{v} = v_k\hat{\mathbf{e}}_k$. Plugging in $f\mathbf{A} = fA_k\hat{\mathbf{e}}_k$ for $\mathbf{v}$, we get

$$\nabla \times (f\mathbf{A}) = \hat{\mathbf{e}}_i\epsilon_{ijk}\partial_j(fA_k) = \hat{\mathbf{e}}_i\epsilon_{ijk}(\partial_j f)A_k + \hat{\mathbf{e}}_i\epsilon_{ijk}f\partial_j A_k.$$

The first term we recognize as a cross product of $\nabla f = \hat{\mathbf{e}}_j\partial_j f$ and $\mathbf{A} = A_k\hat{\mathbf{e}}_k$, and we can pull f out of the second term and recognize the rest as the curl of $\mathbf{A}$. Thus the whole expression becomes

$$\nabla f \times \mathbf{A} + f \nabla \times \mathbf{A}.$$

(c) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot \nabla \times \mathbf{B}$

Solution: The cross product $\mathbf{A} \times \mathbf{B}$ is $\hat{\mathbf{e}}_i\epsilon_{ijk}A_jB_k$ in Einstein notation. Dotted with $\nabla = \hat{\mathbf{e}}_i\partial_i$, we get

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \partial_i(\epsilon_{ijk}A_jB_k) = \epsilon_{ijk}(\partial_i A_j)B_k + \epsilon_{ijk}A_j\partial_i B_k$$

Renaming the dummy indices $i \mapsto j \mapsto k \mapsto i$ in the first term, we can rewrite it as $B_i\epsilon_{jki}\partial_j A_k$. Since ϵ_{ijk} is invariant under a cyclic permutation of its indices, we get $B_i\epsilon_{ijk}\partial_j A_k$. The last three factors now can be recognized as the i -th component of $\nabla \times \mathbf{A}$, so the whole term is $\mathbf{B} \cdot \nabla \times \mathbf{A}$.

Similarly, we can rename the indices $i \leftrightarrow j$, then swap the first two indices on ϵ_{jik} using the antisymmetry of the Levi-Civita symbol to get $\epsilon_{ijk}A_j\partial_i B_k = \epsilon_{jik}A_i\partial_j B_k = -A_i\epsilon_{ijk}\partial_j B_k$. Again, we recognize $\epsilon_{ijk}\partial_j B_k$ as the i -th component of $\nabla \times \mathbf{B}$. Therefore, the second term in the expansion of $\nabla \cdot (\mathbf{A} \times \mathbf{B})$ is $-\mathbf{A} \cdot \nabla \times \mathbf{B}$.

(d) $\nabla(f^n) = n f^{n-1} \nabla f$, for any real n , it need not be an integer.

Solution: We prove this by computing one component of the gradient: if it's true for an arbitrary component then it's true for the whole vector (gradient). The proof then just follows from the chain rule for partial derivatives.

$$[\nabla(f^n)]_i = \frac{\partial}{\partial x_i}(f^n) = n f^{n-1} \frac{\partial}{\partial x_i} f = n f^{n-1} [\nabla f]_i$$

(e) $\nabla^2(fg) = f \nabla^2 g + 2(\nabla f) \cdot (\nabla g) + (\nabla^2 f) g$

Solution: We can evaluate $\nabla^2(fg)$ using the definition of ∇^2 as $\nabla \cdot \nabla$ and the product rules of $\nabla \cdot$ and ∇ . Therefore,

$$\begin{aligned} \nabla^2(fg) &= \nabla \cdot (\nabla(fg)) = \nabla \cdot [f \nabla g + (\nabla f) g] \\ &= (\nabla f) \cdot (\nabla g) + f \nabla \cdot \nabla g + (\nabla \cdot \nabla f) g + (\nabla f) \cdot (\nabla g) = f \nabla^2 g + 2(\nabla f) \cdot (\nabla g) + (\nabla^2 f) g. \end{aligned}$$

2 Gauge transformation

In electromagnetism, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be expressed as derivatives of the scalar potential ϕ and the vector potential $\mathbf{A}$,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1)$$

Show that changing ϕ to $\phi - \frac{\partial \lambda}{\partial t}$ and $\mathbf{A}$ to $\mathbf{A} + \nabla \lambda$ does not change the value of the electric and magnetic fields.

Solution: Plugging the transformed vector potential $\mathbf{A} + \nabla \lambda$ into $\nabla \times \mathbf{A}$, we get $\nabla \times \mathbf{A} + \nabla \times \nabla \lambda$. Since the curl of a gradient is zero, the second term is zero, so we get the same magnetic field.

On the other hand, the gradient of $\phi - \frac{\partial \lambda}{\partial t}$ has an extra term $-\nabla(\frac{\partial \lambda}{\partial t})$, while the time derivative of $\mathbf{A}$ has an extra term $\frac{\partial}{\partial t}(\nabla \lambda)$. Since the partial derivatives ∂_i and ∂_t commute, these two terms are opposites of each other and will cancel. Thus the electric field derived from the transformed potentials is also the same.

3 Complex analytic functions and vector fields

Recall Cauchy's integral theorem for complex analytic functions,

Theorem (Cauchy). *If $f(z)$ is analytic on a simply connected domain U enclosed by a closed contour C , then*

$$\oint_C f(z) dz = 0.$$

There is a similar theorem for irrotational vector fields, i.e. vector fields that have zero curl,

Theorem (Green's). *If $\mathbf{F}$ is a two-dimensional vector field such that $\nabla \times \mathbf{F} = 0$ on U , a simply connected domain enclosed by a closed contour C , then*

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = 0.$$

We will see that the similarity between the two theorems is not a coincidence.

- (a) Let u and v be the real and imaginary parts of $f(x + iy) = u(x, y) + iv(x, y)$. Show that if f is analytic, the vector fields $\mathbf{F}(x, y) = (u(x, y), -v(x, y))$ and $\mathbf{G}(x, y) = (v(x, y), u(x, y))$ have zero curl.

Solution: Recall the Cauchy–Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

For a two-dimensional vector field $A = (A_x, A_y)$, the curl is equal to $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$. Thus

$$\nabla \times \mathbf{F} = \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

and

$$\nabla \times \mathbf{G} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

- (b) Show that $f(z) dz = (\mathbf{F} + i\mathbf{G}) \cdot d\mathbf{l}$, where $d\mathbf{l}$ is the line element along a contour γ . (Hint: use a parametrization $z(t) = x(t) + iy(t)$ to compute both sides of the identity explicitly.)

Solution: With the parametrization $z(t) = x(t) + iy(t)$, the differential dz is $(\dot{x} + i\dot{y}) dt$. Thus

$$f(z) dz = (u + iv)(\dot{x} + i\dot{y}) dt = [(u\dot{x} - v\dot{y}) + i(v\dot{x} + u\dot{y})] dt.$$

On the other hand, since the line element $d\mathbf{l}$ along the contour γ is $(\dot{x}, \dot{y}) dt$, we get

$$(\mathbf{F} + i\mathbf{G}) \cdot d\mathbf{l} = (u, -v) \cdot (\dot{x}, \dot{y}) dt + i(v, u) \cdot (\dot{x}, \dot{y}) dt = [(u\dot{x} - v\dot{y}) + i(v\dot{x} + u\dot{y})] dt.$$

Now we see that the two sides of the identity are equal.

- (c) Use Green's theorem to prove Cauchy's integral theorem.

Solution: We have shown in part (a) that $\mathbf{F}$ and $\mathbf{G}$ are both irrotational. Thus by Green's theorem, $\oint_C \mathbf{F} \cdot d\mathbf{l} = 0$ and $\oint_C \mathbf{G} \cdot d\mathbf{l} = 0$. This means that

$$\oint_C f(z) dz = \oint_C (\mathbf{F} + i\mathbf{G}) \cdot d\mathbf{l} = \oint_C \mathbf{F} \cdot d\mathbf{l} + i \oint_C \mathbf{G} \cdot d\mathbf{l} = 0$$

4 Diffusion

Very often, if there is high density of something in some part of a physical system and low density somewhere else, things will flow from the region of high density to the region of low density. For small differences, it's common to assume the form

$$\mathbf{J} = -k\nabla\rho, \quad k > 0.$$

This is called Fick's law of diffusion.

- (a) Prove that in this case the flow is irrotational, i.e. $\nabla \times \frac{\mathbf{J}}{\rho} = 0$. (Hint: use the product rules proved in question 1.)

Solution: Applying the product rule 1(b) to $f = \frac{1}{\rho}$ and $\mathbf{A} = \mathbf{J}$, we get

$$\nabla \times \frac{\mathbf{J}}{\rho} = -k\nabla\left(\frac{1}{\rho}\right) \times \nabla\rho - \frac{k}{\rho}(\nabla \times \nabla\rho).$$

Applying the product rule 1(d), $\nabla\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\nabla\rho$, so the first term becomes $\frac{k}{\rho^2}\nabla\rho \times \nabla\rho$. The cross product of a vector with itself is zero, so this term is zero. The second term also goes to zero, because the curl of a gradient is zero. This completes the proof that $\nabla \times \frac{\mathbf{J}}{\rho} = 0$.

- (b) Prove that if the density is conserved, i.e. if it satisfies the continuity equation $\partial_t\rho + \nabla \cdot \mathbf{J} = 0$, it moves diffusively, i.e.

$$\partial_t\rho = D\nabla^2\rho$$

for some constant D .

Solution: The continuity equation for the flow $\mathbf{J} = -k\nabla\rho$ takes the form

$$\partial_t\rho + \nabla \cdot (-k\nabla\rho) = 0$$

Recognizing that $\nabla \cdot \nabla\rho$ is just the Laplacian $\nabla^2\rho$, we see that the second term is $-k\nabla^2\rho$. Moving this term to the other side, we get

$$\partial_t\rho = k\nabla^2\rho.$$

In this case, the diffusion constant D is just the coefficient k .

- (c) By plugging the following form for ρ

$$\rho(x, t) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{x^2}{2Dt}\right)$$

show that it satisfies the diffusion equation ($D > 0$ is a positive constant). Sketch $\rho(t)$ for a few values of time $t > 0$ to visualize the “spreading out” behaviour of this function.

Solution: For the given ρ , the left hand side of the diffusion equation evaluates to

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{2\pi Dt}} \right) \exp\left(-\frac{x^2}{2Dt}\right) + \frac{1}{\sqrt{2\pi Dt}} \frac{\partial}{\partial t} \exp\left(-\frac{x^2}{2Dt}\right) \\ &= -\frac{\pi D}{(2\pi Dt)^{3/2}} \exp\left(-\frac{x^2}{2Dt}\right) + \frac{1}{\sqrt{2\pi Dt}} \left(\frac{x^2}{2Dt^2} \right) \exp\left(-\frac{x^2}{2Dt}\right) \\ &= \frac{D(x^2 - Dt)}{2\sqrt{2\pi}(Dt)^{5/2}} \exp\left(-\frac{x^2}{2Dt}\right). \end{aligned}$$

The right hand side is just $D\partial_x^2\rho$, so first we compute

$$\frac{\partial\rho}{\partial x} = -\frac{x}{\sqrt{2\pi}(Dt)^{3/2}} \exp\left(-\frac{x^2}{2Dt}\right).$$

Next the second order derivative is

$$\begin{aligned} \frac{\partial^2\rho}{\partial x^2} &= -\frac{1}{\sqrt{2\pi}(Dt)^{3/2}} \exp\left(-\frac{x^2}{2Dt}\right) + \frac{x^2}{\sqrt{2\pi}(Dt)^{5/2}} \exp\left(-\frac{x^2}{2Dt}\right) \\ &= \frac{x^2 - Dt}{\sqrt{2\pi}(Dt)^{5/2}} \exp\left(-\frac{x^2}{2Dt}\right) \end{aligned}$$

Now it is easy to see that $\partial_t\rho = \frac{D}{2}\partial_x^2\rho$.