

1 Kronecker delta, Levi-Civita symbol, and Einstein summation convention

In vector calculus, we need to manipulate expressions containing objects with multiple components. Here are three notations that will simplify these types of calculations.

- The **Kronecker delta** is an object with two indices, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1)$$

It is *symmetric*, i.e., swapping its indices doesn't change its value,

$$\delta_{ij} = \delta_{ji}. \quad (2)$$

- The **Levi-Civita symbol** has three indices, and is defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312, \\ -1 & \text{if } ijk = 132, 213, 321, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

The “otherwise” case includes all cases where at least two of the indices are equal,

$$\epsilon_{111} = \epsilon_{112} = \epsilon_{122} = \cdots = 0. \quad (4)$$

The Levi-Civita symbol is *antisymmetric* (also called alternating), i.e., swapping any two of its indices changes its sign,

$$\epsilon_{ijk} = -\epsilon_{jik}, \quad \epsilon_{ijk} = -\epsilon_{ikj}, \quad \epsilon_{ijk} = -\epsilon_{kji}. \quad (5)$$

- **Einstein summation convention**, or Einstein notation, is a notational convention that says *whenever an index variable appears twice in a single term, it implies a summation of that term over all the values of that index*. For example, an expression like

$$v = \sum_{i=1}^3 v_i \mathbf{e}_i = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \quad (6)$$

is simplified to

$$v = v_i \mathbf{e}_i. \quad (7)$$

In Einstein notation, an index can be repeated at most twice in every term. A repeated index, that is, an index that is summed over is called a *dummy index*. An index that is not repeated is called a *free index*. The free indices in every term of an expression and on both sides of an equation must match.

Quick exercise: Which of the following are allowed in Einstein summation convention?
Warm-Up:

$$v_i = u_i, \quad v_i = v_j, \quad u_i v_i = u_j u_j, \quad u_i u_i v_i v_i$$

$$1. a_i b_j b_k; \quad 2. u_i v_i w_j - u_i v_j w_i + u_j v_i w_i; \quad 3. x_{ab} y_{bc} z_{cd} = -x_{ab} y_{ac} z_{ad}; \quad 4. \delta_{ik} \delta_{jl} s_{ij} t_{kl} = \eta_{abcd} p_{ab} q_{cd}.$$

Combining these notational conventions, we have the following expressions for some common vector operations. Prove these expressions by expanding the sums and listing the components.

$$(a) \mathbf{u} \cdot \mathbf{v} = \delta_{ij} u_i v_j = u_i v_i$$

$$(b) (\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} u_j v_k$$

$$(c) \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \epsilon_{ijk} u_i v_j w_k$$

2 Symmetry of Kronecker delta and the Levi-Civita symbol

Use the symmetry of the Kronecker delta and the Levi-Civita symbol to show the following facts,

- (a) Suppose M_{ij} is an object that satisfies $M_{ij} = M_{ji}$, show that $\epsilon_{ijk} M_{jk} = 0$.
- (b) Using the expression given in the previous problem for the dot product and the cross product, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

3 Relations between Kronecker delta and Levi-Civita symbol

Check that the following identities hold,

$$(a) \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

$$(b) \epsilon_{ijk} \epsilon_{ijm} = 2\delta_{km}$$

$$(c) \epsilon_{ijk} \epsilon_{ijk} = 6$$

4 Vector identities

- (a) Prove the Jacobi identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0 \quad (8)$$

- (b) Prove double vector product identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad (9)$$