

1 Analytic or not?

Which of the following functions are analytic? What are their domains of analyticity? Try to use the easiest criterion (Cauchy-Riemann condition, independence from z^* , composition of analytic functions).

(a) $f(x + iy) = x^3 - 3xy^2 + i(3x^2y - y^3)$

Solution: Since both the real and imaginary parts of this function are polynomials, it's easiest to use Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, so the function is analytic. The domain of analyticity is the entire complex plane.

However, the solution is even more straightforward if one realizes that

$$f(x + iy) = x^3 - 3xy^2 + i(3x^2y - y^3) = (x + iy)^3 = z^3, \quad (1)$$

in which case it is easy to see that the function is analytic in the entire complex plane.

(b) $f(x + iy) = \frac{x - iy}{x^2 + y^2}$

Solution: We can recognize the expressions $x - iy$ and $x^2 + y^2$ as $\bar{z}$ and $z\bar{z}$, respectively, so it is easier to rewrite f in terms of z and $\bar{z}$,

$$f(z, \bar{z}) = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

Since this expression does not depend on $\bar{z}$, we see that f is analytic except at $z = 0$. Note that canceling $\bar{z}$ does not run the risk of division by zero, because $\bar{z}$ is zero if and only if z is zero, and this singularity is already accounted for.

(c) $f_{\pm}(x + iy) = e^x(x \cos y \pm y \sin y) + ie^x(x \sin y \mp y \cos y)$

Solution: Since $\cos y$ and $\sin y$ appear in the real and imaginary parts of the function, respectively, we may want to try applying Euler's formula to convert them into an exponential factor. With that in mind, we can expand the sum and collect them according to the $e^x x$ and $ie^x y$ factors

$$\begin{aligned} e^x(x \cos y \pm y \sin y) + ie^x(x \sin y \mp y \cos y) &= e^x x \cos y + ie^x x \sin y \pm e^x y \sin y \mp ie^x y \cos y \\ &= e^x x(\cos y + i \sin y) \mp ie^x y(\cos y + i \sin y) \end{aligned}$$

Applying Euler's formula, we get

$$e^x x e^{iy} \mp ie^x y e^{iy} = e^{x+iy}(x \mp iy) = \begin{cases} e^z \bar{z} & \text{for } f_+ \\ e^z z & \text{for } f_- \end{cases}.$$

Now we can see clearly that f_+ is not analytic and f_- is.

2 Singularities

Find the singularities of the following functions and describe them. If they are simple poles, compute the residue. If it has a branch point, choose a branch cut.

(a) $\frac{1}{z^4 + z^2}$

Solution: The function is a rational function, so it can only have poles coming from the roots of the denominator. We can rewrite the expression as

$$\frac{1}{z^2(z+1)} = \frac{1}{z^2(z+i)(z-i)},$$

which shows that we have a 2nd order pole at $z = 0$, and simple poles at $z = i$ and $z = -i$.

To find the residue at $z = i$, we can simply multiply the expression by $(z - i)$ and evaluate the resulting expression at $z = i$:

$$\frac{1}{z^2(z+i)(z-i)}(z-i) \Big|_{z=i} = \frac{1}{z^2(z+i)} \Big|_{z=i} = \frac{1}{(i)^2(2i)} = \frac{i}{2}.$$

A similar procedure can be done for the simple pole at $z = -i$ to yield a residue of $-i/2$.

(b) $\frac{\pi}{\sin(\pi z)}$

Solution: The singularities of this function come from the zeros of the denominator. Solving $\sin(\pi z) = 0$ gives us the solutions $z = \dots, -2, -1, 0, 1, 2, \dots$, that is, $z_n = n$ for all integers n .

We can find the residues at these points by first starting at $z = 0$. Taylor expanding $\sin(\pi z)$, we get

$$\frac{\pi}{\pi z - \frac{(\pi z)^3}{3!} + \dots} = \frac{1}{z(1 - \frac{(\pi z)^2}{3!} + \dots)}.$$

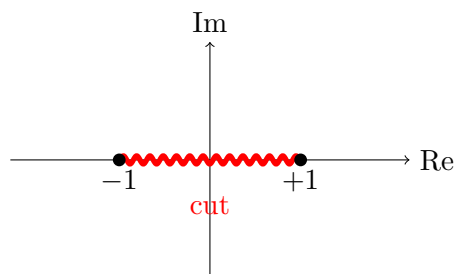
Here it is straightforward to see that to find the residue at $z = 0$ we can multiply the expression by z and evaluate the resulting expression at $z = 0$, which yields 1 as the residue.

To find the residues at the other points we can note that the procedure is similar, except we are just expanding $\sin(\pi(z+n)) = \sin(\pi z + n\pi)$ at $z = 0$. For even n , we are expanding $\sin(\pi z + n\pi) = \sin(\pi z)$, which means the residue should be 1 as we found above. For odd n , we are instead expanding $\sin(\pi z + n\pi) = -\sin(\pi z)$, which means that we would get residues of -1 .

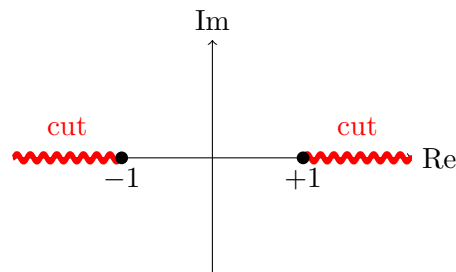
(c) $\sqrt{z^2 - 1}$

Solution: Since the function contains a square root, it will have branch points whenever the argument of the square root goes to zero. These branch points are $z_1 = 1$ and $z_2 = -1$. A valid branch cut is the line segment along the real axis from -1 to 1 , connecting the two branch points. Other branch cuts are also valid, such as line segments from the branch points extending out to infinity; as long as there is no way to wrap around a single branch point, we can keep the function single-valued.

Option 1: Cut connecting branch points



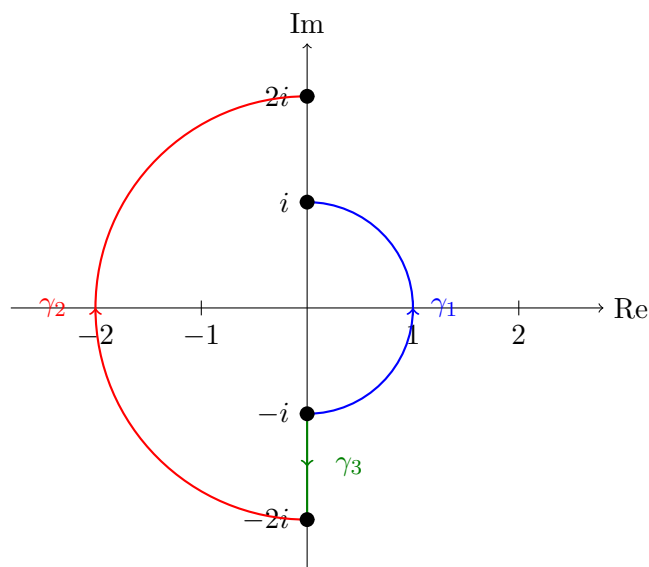
Option 2: Cuts to infinity



3 Contour integrals

Evaluate the integral $\int_{\gamma} \frac{1}{z} dz$ along the following contours.

For reference, below are a visualization of the contours described in the following parts:



- (a) γ_1 is the semicircle from $-i$ to i , traversed counterclockwise.

Solution: This arc can be parametrized by $z(t) = e^{it}$ with $t \in [-\pi/2, \pi/2]$. Thus $dz = ie^{it} dt$.

Plugging this into the integral, we get

$$\int_{\gamma_1} \frac{1}{z} dz = \int_{-\pi/2}^{\pi/2} \frac{1}{e^{it}} ie^{it} dt = i \int_{-\pi/2}^{\pi/2} dt = \boxed{i\pi}.$$

- (b) γ_2 is the semicircle from $-2i$ to $2i$, traversed clockwise.

Solution: Similar to the previous contour, this semicircle can be parametrized by $z(t) = 2e^{-it}$, where $t \in [\pi/2, 3\pi/2]$. Note that t has a negative coefficient in the exponent in the parametrization. This is necessary since the arc is traversed clockwise, so the phase angle should decrease as t increases.

Now $dz = -2ie^{-it} dt$, so the integral becomes

$$\int_{\gamma_2} \frac{1}{z} dz = \int_{\pi/2}^{3\pi/2} \frac{1}{2e^{-it}} (-2i) e^{-it} dt = -i \int_{\pi/2}^{3\pi/2} dt = \boxed{-i\pi}$$

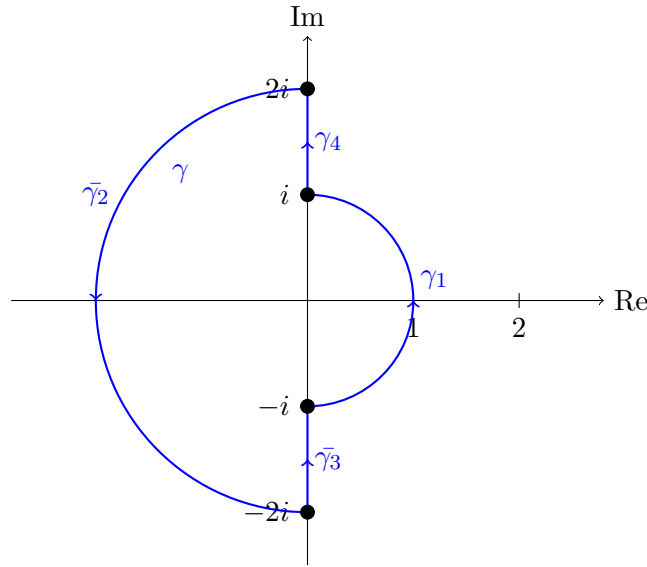
(c) γ_3 is the line segment $-i$ to $-2i$

Solution: The line segment can be parametrized as $z(t) = (-i)(1-t) + (-2i)t = -it - i$, where $t \in [0, 1]$. With $dz = -i dt$, we can evaluate the integral as

$$\int_{\gamma_3} \frac{1}{z} dz = \int_0^1 \frac{1}{-it-i} (-i) dt = \int_0^1 \frac{1}{t+1} dt = \ln(t+1) \Big|_{t=0}^1 = \boxed{\ln 2}$$

(d) *Bonus question:* Using the above results, compute the integral along the closed contour: $-i$ to i along a counterclockwise semicircle, then to $2i$ along a line segment, then to $-2i$ along a counterclockwise semicircle, finally back to $-i$ along a line segment.

Solution: Below is a visualization of the closed contour integral:



Among the four segments, the only one we haven't integrated along is the segment γ_4 from i to $2i$, but this contour is just γ_3 transformed under $z \mapsto -z$. Since $\int_{\gamma} \frac{1}{-z} d(-z) = \int_{\gamma} \frac{1}{z} dz$, this integral is equal to the integral along γ_3 . Now we can evaluate the total integral

$$\oint_{\gamma} \frac{1}{z} dz = \int_{\gamma_1} \frac{1}{z} dz + \int_{\gamma_2} \frac{1}{z} dz + \int_{\gamma_3} \frac{1}{z} dz + \int_{\gamma_4} \frac{1}{z} dz = i\pi + \ln 2 - (-i\pi) - \ln 2 = \boxed{2\pi i}.$$

This answer should make sense as $1/z$ has a simple pole at $z = 0$ with a residue of 1, and the Residue Theorem tells us that a closed contour integral would yield $2\pi i$ times the sum of the residues of all singularities inside the contour.