

1 Arithmetic of complex numbers

Simplify the following complex numbers into the Cartesian form $a + ib$. Find the modulus, phase angle, and complex conjugate. Note that some expressions may be easier to evaluate if written in polar form: $z = a + ib = |z|e^{i\phi}$, where $|z| = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$.

(a) $(1 - 2i)(3 + i)$

(b) $(1 + i)^6$

(c) $\frac{1 + 2i}{4i - 3}$

(d) $\ln(1 + \sqrt{3}i)$

2 Using complex numbers to do integrals

In a previous discussion, we computed the integral

$$I = \int_0^\infty e^{-\alpha x} \sin x \, dx \quad \text{with } \alpha > 0$$

using integration by parts.

There is another approach. Think of $\sin x$ as the imaginary part of e^{ix} , so the integral is also equal to

$$I = \text{Im} \left(\int_0^\infty e^{-\alpha x} e^{ix} \, dx \right).$$

Now integrate this function and take the imaginary part.

Challenge: There is yet another approach based on Taylor series. Expand $\sin x$ as a power series and integrate term by term. You will need the factorial function from homework 2 and the geometric series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n.$$

(Hint to get you started: The integral of a summation is the summation of integrals.)

3 Another application of using complex numbers to do integrals

Using the substitution $x = \tan u$, we can compute the following integral

$$I = \int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$$

This integral can also be done using complex numbers.

- (a) Find the partial fraction decomposition of $\frac{1}{1+x^2}$ using $1+x^2 = (1+ix)(1-ix)$.
- (b) Evaluate the resulting integral.
- (c) *Challenge:* Use the following fact to show that the result from part (b) is equal to $\tan^{-1} x$,

$$\tan x = i \frac{1 - e^{2ix}}{1 + e^{2ix}} .$$