

Discussion 2 Solutions

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February 2, 2026.

1 Integration Exercises

Evaluate the following integrals.

(a) $\int_0^\infty \frac{r^3}{(r^2+a^2)^3} dr$

Solution: Let $u = r^2$ then $du = 2r dr$. The original integral can be rewritten as

$$\frac{1}{2} \int_0^\infty \frac{r^2(2r dr)}{(r^2+a^2)^3} = \frac{1}{2} \int_0^\infty \frac{u du}{(u+a^2)^3}$$

Next we need to use the partial fraction decomposition,

$$\frac{A}{(u+a^2)^3} + \frac{B}{(u+a^2)^2} + \frac{C}{u+a^2} = \frac{(A+Ba^2+Ca^4) + (B+2Ca^2)u + Cu^2}{(u+a^2)^3}$$

Matching the denominator to u , we see that $C = 0$, $B = 1$, and $A = -a^2$. Thus

$$\begin{aligned} \frac{1}{2} \int_0^\infty \frac{u du}{(u+a^2)^3} &= -\frac{a^2}{2} \int_0^\infty \frac{du}{(u+a^2)^3} + \frac{1}{2} \int_0^\infty \frac{du}{(u+a^2)^2} \\ &= -\frac{a^2}{2} \left(-\frac{1}{2}\right) \left(\frac{1}{(u+a^2)^2}\right) \Big|_{u=0}^\infty + \frac{1}{2}(-1) \left(\frac{1}{u+a^2}\right) \Big|_{u=0}^\infty \\ &= -\frac{1}{4a^2} + \frac{1}{2a^2} = \frac{1}{4a^2} \end{aligned}$$

(b) $\int \frac{dx}{\sqrt{b^2+x^2}}$ (Recall that $1 + \sinh^2 x = \cosh^2 x$ and $\frac{d}{dx} \sinh x = \cosh x$.)

Solution: Take $x = b \sinh u$, then $dx = b \cosh u du$. Also $b^2 + x^2 = b^2(1 + \sinh^2 u) = b^2 \cosh^2 u$. Plugging these into the integral, we get

$$\int \frac{dx}{\sqrt{b^2+x^2}} = \int \frac{b \cosh u du}{\sqrt{b^2 \cosh^2 u}} = \int du = u + C$$

Since $u = \sinh^{-1} \frac{x}{b}$ the final result is

$$\sinh^{-1} \frac{x}{b} + C.$$

(c) $\int_0^\infty e^{-\alpha x} \sin x \, dx$ (Use integration by parts twice.)

Solution: Take $u = e^{-\alpha x}$ and $v = -\cos x$ then $du = -\alpha e^{-\alpha x} dx$ and $dv = \sin x \, dx$. Thus

$$I \equiv \int_0^\infty e^{-\alpha x} \sin x \, dx = (e^{-\alpha x}(-\cos x)) \Big|_0^\infty - \int_0^\infty (-\alpha e^{-\alpha x})(-\cos x) dx = 1 - \alpha \int_0^\infty e^{-\alpha x} \cos x \, dx.$$

Now take $u = e^{-\alpha x}$ again, and take $w = \sin x$, so that $dw = \cos x \, dx$. Thus

$$I = 1 - \alpha \left((e^{-\alpha x} \sin x) \Big|_0^\infty - \int_0^\infty (-\alpha e^{-\alpha x}) \sin x \, dx \right) = 1 - \alpha^2 I.$$

Now we have an equation $I = 1 - \alpha^2 I$ from which we can solve for I . Thus

$$I = \frac{1}{1 + \alpha^2}$$

2 Gaussian Integrals

The Gaussian is an extremely prevalent function in physics, so it's a good idea to get to know it better.

Recall that

$$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}.$$

Use this to prove that

$$\int_{-\infty}^\infty e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}.$$

(Hint: Complete the square.)

Solution: Completing the square gives us

$$-ax^2 + bx + c = -a \left(x - \frac{b}{2a} \right)^2 + c + \frac{b^2}{4a}$$

This means that if we take $u = \sqrt{a} \left(x - \frac{b}{2a} \right)$, then $dx = \frac{1}{\sqrt{a}} du$, and $u^2 = a \left(x - \frac{b}{2a} \right)^2$. Thus

$$\int_{-\infty}^\infty e^{-a \left(x - \frac{b}{2a} \right)^2 + c + \frac{b^2}{4a}} dx = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}+c} \int_{-\infty}^\infty e^{-u^2} du$$

The remaining integral is just the usual Gaussian integral, which is equal to $\sqrt{\pi}$. Substituting this in gives us the desired result.

3 Leibniz Rule: Differentiating under the Integral Sign

The Leibniz integral rule for differentiation under the integral sign states that if an integrand depends on some extra parameter, we can differentiate the integral by taking a partial derivative of the integrand, that is

$$\frac{d}{dx} \left(\int_a^b f(x, t) \, dt \right) = \int_a^b \frac{\partial}{\partial x} f(x, t) \, dt.$$

Use this fact to evaluate the integral

$$I = \int_0^1 \frac{t - 1}{\ln t} \, dt$$

by thinking of it as a special case of the integral

$$I(\alpha) = \int_0^1 \frac{t^\alpha - 1}{\ln t} \, dt.$$

(a) Take the α derivative of both sides.

Solution:

$$I'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha} \left(\frac{t^\alpha - 1}{\ln t} \right) dt = \int_0^1 \frac{t^\alpha \ln t}{\ln t} dt = \int_0^1 t^\alpha dt.$$

(b) Evaluate the t integral.

Solution: Assuming $\alpha \neq -1$, we have

$$\int_0^1 t^\alpha dt = \left. \frac{t^{\alpha+1}}{\alpha+1} \right|_0^1 = \frac{1}{\alpha+1}$$

(c) Integrate the result with respect to α . (This is an indefinite integral, so remember the integration constant.)

Solution:

$$\int \frac{d\alpha}{\alpha+1} = \int \frac{d(\alpha+1)}{\alpha+1} = \ln(\alpha+1) + C.$$

(d) Determine the integration constant by evaluating $I(0)$.

Solution: Since

$$I(0) = \int_0^1 \frac{t^0 - 1}{\ln t} dt = 0,$$

we must have $(\ln(\alpha + 1) + C)|_{\alpha=0} = C = 0$, so

$$I(\alpha) = \ln(\alpha + 1)$$

(e) Finally, set $\alpha = 1$ to get I .

Solution:

$$I = I(1) = \ln(1 + 1) = \ln 2.$$

4 Additional Questions

4.1 Integration Exercises II

- Integration by Parts: $\int u \, dv = uv - \int v \, du$.
- U-Substitution: $\int f(g(x))g'(x) \, dx = \int f(u) \, du$.

Here are some integrals to warm yourselves up on these techniques,

(a) $\int b^{ax} \, dx$

Solution: Since the integrand is a composite function, we can make a u-substitution with the inner function ax as u and $dx = du/a$. Thus the integral becomes

$$\int b^u \frac{du}{a} = \frac{1}{a} \int b^u du = \frac{b^u}{a \ln b} + C = \frac{b^{ax}}{a \ln b} + C,$$

where we've used the fact that the anti-derivative of b^x is $b^x / \ln b$.

(b) $\int t^a \, dt$

Solution: To evaluate this integral without the power rule, we can take $u = t^a$ and $v' = 1$ (implicit) in the above formula, so that $v = t$. Thus

$$\int t^a \times 1 \, dt = t^a \times t - \int at^{a-1} \times t \, dt = t^{a+1} - a \int t^a \, dt.$$

Moving the second term on the right to the left hand side, we get

$$(a + 1) \int t^a \, dt = t^{a+1}$$

Dividing both sides by $a + 1$ we finally arrive at

$$\int t^a \, dt = \frac{t^{a+1}}{a + 1} + C.$$

(c) $\int \ln x \, dx$

Solution: In the integrand, the factor $\ln x$ simplifies to a power of x after taking the derivative, and the remaining factor 1 has a simple anti-derivative. Thus we should use integration by parts with $u = \ln x$ and $v = x$

$$\int \ln x \times 1 \, dx = x \ln x - \int \frac{1}{x} \times x \, dx = x \ln x - \int dx = x \ln x - x + C.$$

4.2 Leibniz Rule II

Using the Leibniz rule, prove that the integral

$$F(n) = \int_0^\infty x^n e^{-x} \, dx$$

satisfies $F(n) = nF(n-1)$. This time, treat it as a special case of the integral

$$F(n, a) = \int_0^\infty x^n e^{-ax} \, dx.$$

Solution: Taking the derivative of $F(n-1, a)$ with respect to a , we get

$$\frac{d}{da} F(n-1, a) = \frac{d}{da} \int_0^\infty x^{n-1} e^{-ax} \, dx = \int_0^\infty \frac{\partial}{\partial a} (x^{n-1} e^{-ax}) \, dx = - \int_0^\infty x^n e^{-ax} \, dx = -F(n, a).$$

On the other hand, using a substitution $t = ax$, we can express $F(n, a)$ in terms of $F(n)$,

$$F(n, a) = \int_0^\infty x^n e^{-ax} \, dx = a^{-n-1} \int_0^\infty (ax)^n e^{-ax} d(ax) = a^{-n-1} \int_0^\infty t^n e^{-t} \, dt = a^{-n-1} F(n).$$

If we plug this into $\frac{d}{da} F(n-1, a)$, we get

$$\frac{d}{da} (a^{-(n-1)-1} F(n-1)) = \frac{d}{da} (a^{-n} F(n-1)) = -na^{-n-1} F(n-1)$$

and $-F(n, a)$ is simply $-a^{-n-1} F(n)$. Canceling the factor $-a^{-n-1}$ from these two expressions, we get the relation $F(n) = nF(n-1)$.