

1 Dynamics of a vibrating string

Consider a string stretched between two fixed points $x = 0$ and $x = L$. Suppose the string only moves transversely, so that its motion can be described by the transverse displacement $y(x, t)$ at every point x along the string and at any time t . The equation of motion for such a string is

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}. \quad (1)$$

where v is a positive real constant. Because of the occurrence of different partial derivatives, this equation is called a partial differential equation. It is *linear* in the function $y(x, t)$ and of *2nd order* in the derivatives; i.e. a linear 2nd order partial differential equation.

- (a) These equations happen to have very simple forms of general solutions. Verify that $y_1(x, t) = F(x - vt)$ and $y_2(x, t) = G(x + vt)$ solve the equation, where F and G are arbitrary smooth functions.

- (b) Using the ansatz $y(x, t) = u(x) e^{i\omega t}$, we can separate the time-dependent part of the solution from the spatially varying part to obtain an ordinary linear differential equation for $u(x)$. Verify that $u(x)$ satisfies

$$-\frac{d^2 u}{dx^2} = k^2 u \quad (2)$$

where $k = \omega/v$.

- (c) Since the ends of the string are tied down, there can be no motion at the two ends, so $u(0) = u(L) = 0$ in the physical scenario that we are considering. Such constraints are called *boundary conditions*. The set of smooth **real** functions on $[0, L]$ that satisfies these boundary conditions forms a vector space, V . On this vector space, we have an inner product, defined by

$$\langle f|g \rangle = \frac{2}{L} \int_0^L f(x) g(x) dx.$$

Under this inner product, what is the norm of $u(x) = \sin(\pi x/L)$?

- (d) The second order differential operator $\hat{T} = -\frac{\partial^2}{\partial x^2}$ acts on V , i.e, $\hat{T}f = -\frac{\partial^2 f}{\partial x^2}$ for a smooth function $f : [0, L] \rightarrow \mathbb{R}$. Show that this operator is Hermitian. In other words, show that for any function $f, g \in V$,

$$\langle \hat{T}f|g \rangle = \langle f|\hat{T}g \rangle.$$

Hint: integrate by parts.

(e) The differential equation (2) takes the form of an eigenvalue problem for the operator $\hat{T}$,

$$\hat{T}u = k^2u.$$

Show that $u_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, \dots$ are eigenfunctions of $\hat{T}$. What are the corresponding eigenvalues?

2 Using the Cayley–Hamilton theorem

The *Cayley–Hamilton Theorem* states that any matrix A satisfies its own characteristic equation as a matrix equation

$$p_A(A) = 0$$

where $p_A(x) = \det(A - xI)$ is the characteristic polynomial. Let A be the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Its characteristic polynomial is: $p_A(x) = 10 - 13x + 6x^2 - x^3$.

(a) Verify that A satisfies the Cayley–Hamilton theorem: $p_A(A) = 10I - 13A + 6A^2 - A^3 = 0$.

(b) Rearrange the above equation to find an expression for A^{-1} .

Hint: multiply both sides of the equation by A^{-1} , then isolate the term containing A^{-1} .

(c) Compute A^{-1} using the above method and verify that $A^{-1}A = I$.