

1 Dynamics of a vibrating string

- (a) Verify that $y_1(x, t) = F(x - vt)$ and $y_2(x, t) = G(x + vt)$ solve the wave equation.

Solution: Differentiating $y_1(x, t) = F(x - vt)$ with respect to t once, we get

$$\frac{\partial}{\partial t} F(x - vt) = F'(x - vt) \frac{\partial(x - vt)}{\partial t} = -v F'(x - vt).$$

Taking the t derivative again, we get

$$\frac{\partial^2}{\partial t^2} F(x - vt) = -v \frac{\partial}{\partial t} (F'(x - vt)) = v^2 F''(x - vt).$$

Similarly, taking the x derivative twice, we get

$$\frac{\partial^2}{\partial x^2} F(x - vt) = F''(x - vt).$$

Thus both sides of the wave equation evaluate to $v^2 F''(x - vt)$, so y_1 is a solution. The case of $y_2(x, t) = G(x + vt)$ is similar; the sign from the chain rule appears twice and squares to +1, giving both sides equal to $v^2 G''(x + vt)$.

- (b) Verify that $u(x)$ satisfies $-\frac{d^2 u}{dx^2} = k^2 u$ with $k = \omega/v$.

Solution: Substitute the ansatz $y(x, t) = u(x) e^{i\omega t}$ into the wave equation. The time derivatives act only on $e^{i\omega t}$ and the spatial derivatives act only on $u(x)$:

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 u(x) e^{i\omega t}, \quad \frac{\partial^2 y}{\partial x^2} = u''(x) e^{i\omega t}.$$

Plugging into $\partial_t^2 y = v^2 \partial_x^2 y$ and cancelling the common factor $e^{i\omega t}$:

$$-\omega^2 u(x) = v^2 u''(x) \implies -\frac{d^2 u}{dx^2} = \frac{\omega^2}{v^2} u = k^2 u,$$

where $k = \omega/v$, as claimed. ✓

- (c) What is the norm of $u(x) = \sin(\pi x/L)$?

Solution: The norm is the square root of the inner product of u with itself. We first compute

$$\langle u|u \rangle = \frac{2}{L} \int_0^L \sin^2 \frac{\pi x}{L} dx = \frac{2}{L} \int_0^L \frac{1}{2} \left(1 - \cos \frac{2\pi x}{L} \right) dx = \frac{2}{L} \times \frac{L}{2} = 1.$$

From this, we get that $\|u\| = \sqrt{1} = 1$. (The $2/L$ prefactor in the inner product is chosen precisely so that the $\sin(n\pi x/L)$ have unit norm.)

- (d) Show that $\hat{T} = -\frac{\partial^2}{\partial x^2}$ is Hermitian on V .

Solution: The Hermiticity condition $\langle \hat{T}f|g \rangle = \langle f|\hat{T}g \rangle$ with $\hat{T} = -\partial_x^2$ reads

$$-\frac{2}{L} \int_0^L \frac{\partial^2 f}{\partial x^2} g(x) dx = -\frac{2}{L} \int_0^L f(x) \frac{\partial^2 g}{\partial x^2} dx.$$

Cancelling the overall factor of $-2/L$, we need to show $\int_0^L f'' g dx = \int_0^L f g'' dx$.

Integrate the left hand side by parts,

$$\int_0^L \frac{\partial^2 f}{\partial x^2} g(x) dx = \left. \frac{\partial f}{\partial x} g(x) \right|_0^L - \int_0^L \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} dx.$$

Since functions in V vanish at $x = 0$ and $x = L$, the boundary term is zero. Similarly, integrating the right hand side by parts gives

$$\int_0^L f(x) \frac{\partial^2 g}{\partial x^2} dx = f(x) \left. \frac{\partial g}{\partial x} \right|_0^L - \int_0^L \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} dx = - \int_0^L \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} dx,$$

which matches the left hand side. $\checkmark$

- (e) Show that $u_n(x) = \sin(n\pi x/L)$ are eigenfunctions of $\hat{T}$.

Solution: Acting with $\hat{T}$:

$$\hat{T} u_n(x) = -\frac{\partial^2}{\partial x^2} \sin \frac{n\pi x}{L} = -\frac{n\pi}{L} \frac{\partial}{\partial x} \left(\cos \frac{n\pi x}{L} \right) = \left(\frac{n\pi}{L} \right)^2 \sin \frac{n\pi x}{L} = \left(\frac{n\pi}{L} \right)^2 u_n(x).$$

So u_n is an eigenfunction of $\hat{T}$ with eigenvalue

$$\boxed{k_n^2 = \left(\frac{n\pi}{L} \right)^2}.$$

Combined with $k = \omega/v$, this gives the allowed frequencies $\omega_n = n\pi v/L$ — the normal modes of the string.

2 Using the Cayley–Hamilton theorem

- (a) Verify that $p_A(A) = 10I - 13A + 6A^2 - A^3 = 0$.

Solution: First we compute the powers of A ,

$$A^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 11 \\ 0 & -11 & 2 \end{pmatrix}.$$

Using these, we compute

$$10I - 13A + 6A^2 - A^3 = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} - \begin{pmatrix} 26 & 0 & 0 \\ 0 & 26 & 13 \\ 0 & -13 & 26 \end{pmatrix} + \begin{pmatrix} 24 & 0 & 0 \\ 0 & 18 & 24 \\ 0 & -24 & 18 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 11 \\ 0 & -11 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Find an expression for A^{-1} .

Solution: Multiplying both sides of $10I - 13A + 6A^2 - A^3 = 0$ by A^{-1} gives

$$10A^{-1} - 13I + 6A - A^2 = 0.$$

Moving all terms except $10A^{-1}$ to the right hand side and dividing through by 10,

$$A^{-1} = \frac{1}{10}(A^2 - 6A + 13I).$$

(c) Compute A^{-1} and verify that $A^{-1}A = I$.

Solution: Using the powers of A from part (a),

$$A^{-1} = \frac{1}{10} \left(\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix} - \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Verify:

$$A^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} + \frac{1}{5} & \frac{2}{5} - \frac{2}{5} \\ 0 & \frac{2}{5} - \frac{2}{5} & \frac{1}{5} + \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \quad \checkmark$$