

1 Gram–Schmidt process

The Gram–Schmidt process generates an orthonormal set of vectors from a linearly independent set. There is an iterative process. In this problem, we will walk through the process using the following set of vectors as a starting point,

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}.$$

- (a) First, normalize the first vector $\mathbf{v}_1$ to obtain the first vector $\hat{\mathbf{e}}_1$ in the orthonormal set.

- (b) Next, find the projection of the second vector $\mathbf{v}_2$ along the first unit vector, $(\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$.

- (c) To obtain the second element of the orthonormalized set, we first subtract off the component that lies along the direction of $\hat{\mathbf{e}}_1$ to get a vector $\mathbf{w}_2$ that is orthogonal to the first vector. Draw a diagram to represent this procedure.

- (d) Next we normalize $\mathbf{w}_2$ to get $\hat{\mathbf{e}}_2$.

- (e) Repeat steps (b) through (d) for the third vector $\mathbf{v}_3$. Note that we need to find the projection along both $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$.

2 Solving eigenvalue problems

Here we illustrate the process of solving eigenvalue problems with the following matrix,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

- (a) Evaluate the characteristic polynomial $p(\lambda)$ of A , that is, $\det(A - \lambda I)$.
- (b) Find the eigenvalues of A by observing the structure of the matrix and finding the roots of the characteristic polynomial.
- (c) Choose an eigenvalue and find a corresponding eigenvector.

3 Vector space of $m \times n$ matrices

$m \times n$ matrices with real or complex entries form vector spaces with the operations of matrix addition and scalar multiplication. We can verify that they satisfy the axioms of a vector space.

(a) Check that the addition of $m \times n$ matrices (real or complex) has the following properties

- i. $A + B = B + A$
- ii. $A + (B + C) = (A + B) + C$

(b) What is the zero vector or additive identity in this vector space?

(c) Check that multiplying $m \times n$ matrices by scalars (real or complex) satisfy the following properties

- i. $1 \cdot A = A$
- ii. $(\alpha\beta) \cdot A = \alpha \cdot (\beta \cdot A)$
- iii. $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$
- iv. $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$

As a member of this vector space, an $m \times n$ matrix acts like a vector in $\mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$.