

1 Gram–Schmidt process

- (a) Normalize $\mathbf{v}_1$ to obtain $\hat{\mathbf{e}}_1$.

Solution: The norm of $\mathbf{v}_1$ is $\sqrt{0^2 + 0^2 + 2^2} = 2$, so the normalized vector is

$$\hat{\mathbf{e}}_1 = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- (b) Find $(\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$.

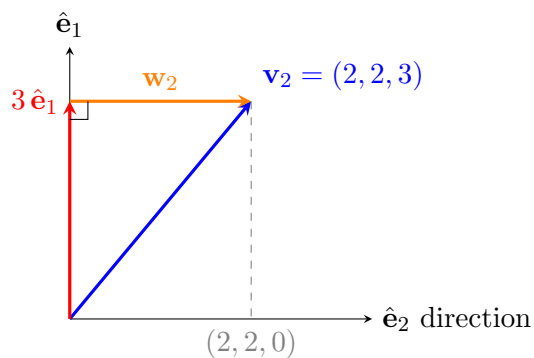
Solution: The dot product is $\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1 = 2 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 = 3$. Thus the component of $\mathbf{v}_2$ along $\hat{\mathbf{e}}_1$ is $3\hat{\mathbf{e}}_1 = [0, 0, 3]^T$.

- (c) Subtract the projection to get $\mathbf{w}_2$.

Solution: After subtracting $(\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$ we get

$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

The diagram below shows $\mathbf{v}_2$, its projection along $\hat{\mathbf{e}}_1$, and the resulting orthogonal vector $\mathbf{w}_2$:



- (d) Normalize $\mathbf{w}_2$ to get $\hat{\mathbf{e}}_2$.

Solution: The norm of $\mathbf{w}_2$ is $\sqrt{2^2 + 2^2 + 0^2} = 2\sqrt{2}$, so

$$\hat{\mathbf{e}}_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(e) Repeat for $\mathbf{v}_3$.

Solution: First, we find the projections of $\mathbf{v}_3$ along $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$,

$$(\mathbf{v}_3 \cdot \hat{\mathbf{e}}_1) \hat{\mathbf{e}}_1 = (-4 \cdot 0 + 0 \cdot 0 + 4 \cdot 1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad (\mathbf{v}_3 \cdot \hat{\mathbf{e}}_2) \hat{\mathbf{e}}_2 = \frac{1}{(\sqrt{2})^2} (-4 \cdot 1 + 0 \cdot 1 + 4 \cdot 0) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}.$$

Subtracting both these projections from $\mathbf{v}_3$, we get

$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \hat{\mathbf{e}}_1) \hat{\mathbf{e}}_1 - (\mathbf{v}_3 \cdot \hat{\mathbf{e}}_2) \hat{\mathbf{e}}_2 = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

Finally, we normalize $\mathbf{w}_3$ to get the third element of the orthonormalized set,

$$\hat{\mathbf{e}}_3 = \frac{1}{\|\mathbf{w}_3\|} \mathbf{w}_3 = \frac{1}{\sqrt{(-2)^2 + 2^2 + 0^2}} \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

2 Solving eigenvalue problems

(a) Evaluate the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Solution: Laplace expanding along the first row, we get

$$\begin{vmatrix} 2 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & -1 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ -1 & 2 - \lambda \end{vmatrix} - 0 + 0 = (2 - \lambda)[(2 - \lambda)^2 + 1] = -\lambda^3 + 6\lambda^2 - 13\lambda + 10.$$

(b) Find the eigenvalues.

Solution: Since A is block diagonal, and one of the blocks is the 1×1 matrix with 2 as its only element, one of its eigenvalues must be $\lambda_1 = 2$. This allows us to factor the characteristic polynomial into $-(\lambda - 2)(\lambda^2 - 4\lambda + 5)$. Thus we can find the roots of the factor $\lambda^2 - 4\lambda + 5$ using the quadratic formula,

$$\lambda_{2,3} = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2} = 2 \pm i.$$

(c) Find an eigenvector.

Solution: Here we demonstrate the process of finding the eigenvector with $\lambda_3 = 2 - i$. To do so, we need to solve the equation $(A - \lambda_3 I)\mathbf{v} = 0$ to find the entries of the vector $\mathbf{v} = [v_1, v_2, v_3]^T$,

$$(A - \lambda_3 I)\mathbf{v} = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} iv_1 \\ iv_2 + v_3 \\ -v_2 + iv_3 \end{bmatrix} = 0.$$

The first equation implies that $v_1 = 0$. The second and third equations are equivalent to each other, and both tell us $v_3 = -iv_2$. Thus the eigenspace of λ_3 is spanned by $[0, 1, -i]^T$.

3 Vector space of $m \times n$ matrices

- (a) Check that matrix addition is commutative and associative.

Solution: The addition of matrices is the entry-wise addition. Thus commutativity and associativity for the addition of matrices follows from these properties of the ordinary addition.

- (b) What is the zero vector?

Solution: The zero vector in this space is the zero matrix whose entries are all zero.

- (c) Check the scalar multiplication properties.

Solution: Again, since the scalar multiplication of matrices is applied entry-wise, these properties follow from the identity, associativity, and distributivity of the ordinary product.

As a member of this vector space, an $m \times n$ matrix acts like a vector in $\mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$.