

1 Pauli matrices

- (a) Show that σ_x , σ_y , σ_z are Hermitian.

Solution: This can be checked easily for σ_z and σ_x , which are real and symmetric. For σ_y , we need only note that $\bar{i} = -i$ and $\overline{(-i)} = i$, so when conjugated, the off-diagonal entries swap, which will be swapped back by the transpose.

- (b) Show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = -i\sigma_x\sigma_y\sigma_z = I$.

Solution: This can be shown by direct computation. Here we prove two of the results.

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + (-i) \cdot i & 0 \cdot (-i) + (-i) \cdot 0 \\ i \cdot 0 + 0 \cdot i & i \cdot (-i) + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$i\sigma_x\sigma_y\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- (c) Prove the commutation relations $[\sigma_x, \sigma_y] = 2i\sigma_z$, $[\sigma_y, \sigma_z] = 2i\sigma_x$, $[\sigma_z, \sigma_x] = 2i\sigma_y$.

Solution: We can show this result using the identity $-i\sigma_x\sigma_y\sigma_z = I$. Multiplying both sides on the right by σ_z , we get $-i\sigma_x\sigma_y\sigma_z^2 = \sigma_z$. Since $\sigma_z^2 = I$, this simplifies to $\sigma_x\sigma_y = i\sigma_z$, where we've also divided the $-i$ to the other side. Taking the conjugate transpose of this, we get $(\sigma_x\sigma_y)^\dagger = (i\sigma_z)^\dagger$. The left hand side expands to $\sigma_y^\dagger\sigma_x^\dagger = \sigma_y\sigma_x$, due to the Hermitian property proven in part (a). Similarly the right hand side simplifies to $-i\sigma_z$. Thus we get $\sigma_y\sigma_x = -i\sigma_z$. Combining this with the previous identity, we get

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = i\sigma_z - (-i\sigma_z) = 2i\sigma_z.$$

The cyclic permutations of the identity $i\sigma_x\sigma_y\sigma_z = I$ are also true, i.e., $i\sigma_y\sigma_z\sigma_x = I$ and $i\sigma_z\sigma_x\sigma_y = I$. They can be proven by left-multiplying σ_x^{-1} (and σ_y^{-1}) then right-multiplying σ_x (and σ_y). Using the permuted versions of this identity, the other commutation relations can be proven.

- (d) Show that $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$.

Solution: In the previous part we have already obtained the intermediate results for the product of distinct Pauli matrices, and we have seen that swapping the order of multiplication adds a minus sign. Thus the sum of the AB and BA products of distinct Pauli matrices must be zero. This corresponds to the $i \neq j$ part of the identity $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$.

The $i = j$ part of the identity is a consequence of the results about the squares of the Pauli matrices in part (b). The anti-commutator of an operator with itself is just twice its square. Since Pauli matrices square to the identity, the anti-commutators of Pauli matrices with themselves are just twice the identity.

2 Matrix functions and series

- (a) Show that $\exp D$ is diagonal with entries $(e^{\lambda_1}, \dots, e^{\lambda_k})$.

Solution: When multiplying diagonal matrices, we simply multiply the diagonal entries, so D^n is the diagonal matrix with entries $(\lambda_1^n, \dots, \lambda_k^n)$. Thus

$$\exp D = \sum_{n=0}^{\infty} \frac{1}{n!} D^n = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_1^n & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_k^n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_k} \end{pmatrix}.$$

- (b) Show the powers of $i\sigma_y$ mod 4.

Solution: Since $\sigma_y^2 = I$, we have $(i\sigma_y)^2 = i^2 \sigma_y^2 = -I$. Thus, $(i\sigma_y)^3 = (i\sigma_y)^2(i\sigma_y) = (-I)(i\sigma_y) = -i\sigma_y$. Then $(i\sigma_y)^4 = (i\sigma_y)^3(i\sigma_y) = (-i\sigma_y)(i\sigma_y) = \sigma_y^2 = I$. Now since $(i\sigma_y)^4 = I$, any power of $i\sigma_y$ can be reduced mod 4 to one of the above four cases. In other words, if $n = 4p + q$, then $(i\sigma_y)^n = (i\sigma_y)^{4p}(i\sigma_y)^q = I^p(i\sigma_y)^q = (i\sigma_y)^q$.

- (c) Show that $\exp(-i\theta \sigma_y)$ is a rotation matrix.

Solution: We write $\exp(-i\theta \sigma_y) = \exp(\theta \cdot (-i\sigma_y))$. The powers of $(-i\sigma_y)$ cycle with period 4: $I, -i\sigma_y, -I, +i\sigma_y$. We can separate the series by congruence classes mod 4,

$$\begin{aligned} \exp(-i\theta \sigma_y) &= \sum_{k=0}^{\infty} \frac{1}{k!} (-i\theta \sigma_y)^k \\ &= \sum_{k=0}^{\infty} \frac{(-i\theta \sigma_y)^{4k}}{(4k)!} + \sum_{k=0}^{\infty} \frac{(-i\theta \sigma_y)^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{(-i\theta \sigma_y)^{4k+2}}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{(-i\theta \sigma_y)^{4k+3}}{(4k+3)!} \\ &= I \sum_{k=0}^{\infty} \frac{\theta^{4k}}{(4k)!} - i\sigma_y \sum_{k=0}^{\infty} \frac{\theta^{4k+1}}{(4k+1)!} - I \sum_{k=0}^{\infty} \frac{\theta^{4k+2}}{(4k+2)!} + i\sigma_y \sum_{k=0}^{\infty} \frac{\theta^{4k+3}}{(4k+3)!}. \end{aligned}$$

Combining the even and odd power sums:

$$= I \cos \theta - i\sigma_y \sin \theta.$$

Now $-i\sigma_y = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so

$$\exp(-i\theta \sigma_y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin \theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Multiplying a vector in the xy -plane by this matrix rotates it counterclockwise by the angle θ .

3 Elementary row operations and Gaussian elimination

- (a) Show that $T_{ij}A$ swaps rows i and j , and $\det T_{ij} = -1$.

Solution: Since each row of T_{ij} only has one non-zero element, each entry of $T_{ij}A$ contains only the contribution from one element of A . Specifically, if we represent the (m, l) -th entry of T_{ij} as t_{ml} and the (l, n) -th entry of A as a_{ln} , then the (m, n) -th entry of $T_{ij}A$ is $\sum_l t_{ml} a_{ln}$. For

m different from i and j , t_{ml} is nonzero only when $l = m$, so the sum reduces to $t_{mm} a_{mn} = a_{mn}$. However if $m = i$, t_{ml} is nonzero only if $l = j$, which reduces the sum to a_{jn} . Similarly, the (j, n) -th entry of $T_{ij}A$ is a_{in} . This shows that the rows i and j of $T_{ij}A$ are exactly the rows j and i of A , respectively.

The determinant of T_{ij} can be computed from the definition of determinants in terms of permutations. Each row of T_{ij} only has one nonzero entry, each occupying a different column, so the only contribution to the sum is the one where we multiply the nonzero entries from every row. This term should carry a negative sign, because the column indices can be obtained by swapping one pair of row indices. Thus $\det T_{ij} = -1$.

- (b) Show that $D_i(k)A$ scales row i by k , and $\det D_i(k) = k$.

Solution: Using the indexing convention from the previous part, the (m, n) -th entry of $D_i(k)A$ contains only one non-zero term, which evaluates to $d_{mm} a_{mn}$. This is equal to a_{mn} if $m \neq i$, but equal to $k a_{mn}$ if $m = i$. Thus the product $D_i(k)A$ has identical rows to A , except for row i , which is multiplied by k .

The determinant of a diagonal matrix is just the product of the diagonal entries, which in this case is k .

- (c) Show that $L_{ij}(r)A$ adds r times row j to row i , and $\det L_{ij}(r) = 1$.

Solution: Again adopting the previous conventions, the (m, n) -th entry of $L_{ij}(r)A$ is equal to $l_{mm} a_{mn} = a_{mn}$ when $m \neq i$. When $m = i$, the sum becomes $l_{ii} a_{in} + l_{ij} a_{jn} = a_{in} + r a_{jn}$. All but the i -th row of $L_{ij}(r)A$ is identical to A . The i -th row of $L_{ij}(r)A$ is the i -th row plus r times the j -th row.

$L_{ij}(r)$ is upper or lower triangular, so the determinant is the product of diagonal entries, which is 1.

- (d) Show that the scaling factors multiply to $\det A$ times $(-1)^s$ where s is the number of row swaps.

Solution: Gaussian elimination of a matrix A can be expressed as multiplying the matrix by a sequence of the row operation matrices above. Since the determinant is multiplicative, $\det A$ is equal to the product of the determinants of the row operation matrices and the final identity matrix. The identity matrix and row addition matrices have determinant 1, so $\det A$ is equal to the product of the determinants of the scaling matrices and the swapping matrices. This is precisely equal to the product of the scaling factors times the parity of the swaps.