

Discussion 1 Solutions

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January 26, 2026.

1 Sketching functions

Sketch these functions, identifying and labeling all important features on your drawing.

1)

$$s(t) = \frac{\sin^3(\omega t)}{t^3}$$

What is the value at $t = 0$? What is the envelope?

Solution: We first notice that $s(t)$ is even:

$$s(-t) = \frac{\sin^3(-\omega t)}{(-t)^3} = \frac{-\sin^3(\omega t)}{-t^3} = s(t).$$

To estimate its behavior around $t = 0$, we Taylor expand $s(t)$

$$s(t) = \frac{1}{t^3} \left(\omega t - \frac{\omega^3 t^3}{6} + \dots \right)^3 \approx \frac{1}{t^3} \left(\omega^3 t^3 - \frac{\omega^3 t^5}{2} \right) = \omega^3 \left(1 - \frac{t^2}{2} \right).$$

Even though the function is strictly undefined at $t = 0$, we see that it looks like a downward-facing parabola with a vertex at ω^3 close to it. Finally, we observe that the zeroes of the function are the zeroes of $\sin(\omega t)$, i.e. $n\pi$ ($n \neq 0$) and its local maxima and minima are respectively at $\pi/2 + 2m\pi$, at which $s(t_{\max}) = 1/t^3$, and $-\pi/2 + 2m\pi$, at which $s(t_{\min}) = -1/t^3$. This is the envelope of the function: when one sketches the function, one should ensure that these extrema do lie on those curves.

2)

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}$$

for $\beta > 0$, $\epsilon > 0$, $\mu > 0$. What does this function look like when ϵ is small? What about when ϵ is large?

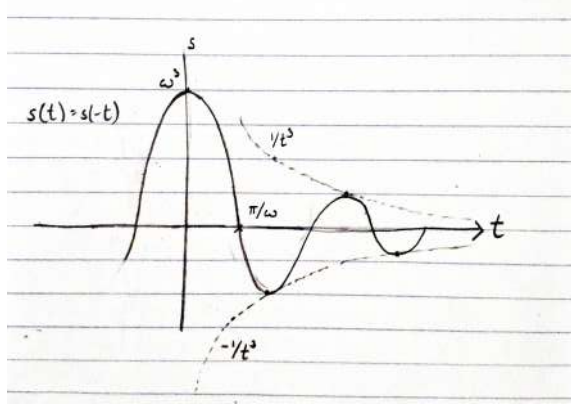


Figure 1: Sketch for Problem 1.1

Solution: Since the domain is restricted to $\epsilon > 0$, so the sketch should not extend to negative values of ϵ . First, we evaluate the function at 0.

$$n(\epsilon = 0) = \frac{1}{e^{-\beta\mu} + 1},$$

By Taylor expanding the function around these three points, we estimate its behavior and can sketch it. Since

$$n'(\epsilon) = -\frac{\beta e^{\beta(\epsilon-\mu)}}{(e^{-\beta\mu} + 1)^2},$$

for small ϵ , the function looks like

$$n(\epsilon) \sim \frac{1}{e^{-\beta\mu} + 1} - \epsilon \cdot \frac{\beta e^{-\beta\mu}}{(e^{-\beta\mu} + 1)^2}.$$

For large ϵ , we see that

$$n(\epsilon) \xrightarrow[\epsilon \rightarrow \infty]{} e^{-\beta\epsilon} \rightarrow 0.$$

Now consider what happens to this function when β is large:

$$\lim_{\beta \rightarrow \infty} n(\epsilon) = \lim_{\beta \rightarrow \infty} \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \begin{cases} 1, & \epsilon < \mu \\ 0, & \epsilon > \mu \end{cases}$$

As suggested by the form of the function, the behavior of the function changes qualitatively at $\epsilon \sim \mu$.

Therefore, the curve is initially slightly less than 1 at $\epsilon = 0$, begins to decrease relatively slowly until $\epsilon \approx \mu$, and decreases exponentially for $\epsilon \gg \mu$.

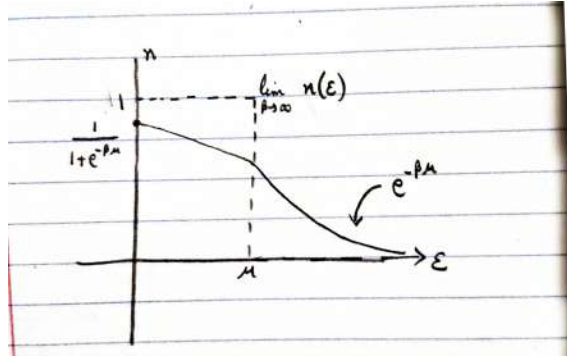


Figure 2: Sketch for Problem 1.2

2 Taylor series

The Taylor expansion of a function $f(x)$ around a point x_0 is

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots \quad (1)$$

1) Find the full Taylor series for $\sin(x)$ around $x = 0$.

Solution: Recall that

$$\begin{aligned} \frac{d}{dx} \sin(x) &= \cos(x), \\ \frac{d}{dx} \cos(x) &= -\sin(x). \end{aligned}$$

So, for $f(x) = \sin(x)$,

$$\begin{aligned} f(x=0) &= \sin(0) = 0, \\ f'(x=0) &= \cos(0) = 1, \\ f''(x=0) &= -\sin(0) = 0, \\ f^{(3)}(x=0) &= -\cos(0) = -1, \\ f^{(4)}(x=0) &= \sin(0) = f^{(0)}(0). \end{aligned}$$

Thus,

$$f^{(n)}(x) \Big|_{x=0} = \begin{cases} 0 & \text{if } n = 0, 2 \pmod{4}, \\ 1 & \text{if } n = 1 \pmod{4} \\ -1 & \text{if } n = 3 \pmod{4}. \end{cases}$$

Therefore, the Taylor expansion of $\sin(x)$ around $x = 0$ is given by

$$\begin{aligned}\sin(x) &= \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1} - \frac{1}{(4k+3)!} x^{4k+3} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.\end{aligned}$$

2) Show that $e^{ix} = \cos(x) + i \sin(x)$ by series expansion around $x = 0$.

Solution: Since $\frac{d}{dx}e^x = e^x$ and $e^0 = 1$, the Taylor expansion around $x = 0$ is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \implies e^{ix} = \sum_{k=0}^{\infty} \frac{i^k x^k}{k!}.$$

Since $i^2 = -1$, we can write this as the sum of a real part and an imaginary part:

$$e^{ix} = \left(\sum_{k=0}^{\infty} \frac{1}{(4k)!} x^{4k} - \frac{1}{(4k+2)!} x^{4k+2} \right) + i \left(\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1} - \frac{1}{(4k+3)!} x^{4k+3} \right)$$

using $i^3 = -i$ and $i^4 = 1$. We recognize that the imaginary part is just the expansion of $\sin(x)$ we derived earlier! The real part is also clearly the derivative of the imaginary part. Since $\frac{d}{dx} \sin(x) = \cos(x)$, we conclude that $e^{ix} = \cos(x) + i \sin(x)$.

3) Compute the first two terms of $\sqrt{1-x^2}$ around $x = 0$ and $x = 1$. What can go wrong here? It may be helpful to sketch the function.

Solution: First compute

$$\begin{aligned}\frac{d}{dx} \sqrt{1-x^2} &= -\frac{x}{\sqrt{1-x^2}} \\ \frac{d^2}{dx^2} \sqrt{1-x^2} &= -x^2(1-x^2)^{-3/2} - (1-x^2)^{-1/2}\end{aligned}$$

Around $x = 0$, the Taylor series up to two terms is

$$f(x=0) + \frac{x^2}{2} f''(x=0) = 1 - \frac{x^2}{2},$$

since $f'(0) = 0$. But at $x = 1$, $-\frac{x}{\sqrt{1-x^2}}$ is undefined! By sketching the function, which is just a semicircle, which has a vertical tangent at $x = 1$. The function does not have a Taylor series expansion at $x = 1$ since its derivative has a singularity there.

3 Additional Questions

3.1 Sketching Functions

Sketch these functions, identifying and labeling all important features on your drawing.

1)

$$E(k) = \sqrt{\frac{k^4}{4m^2} + c^2 k^2}$$

Make sure the curve has the correct shape for small and large k .

Solution: Notice that $E(k) = E(-k)$, so the function is even, so we only need to analyze it for $k \geq 0$. We then expand it around $k = 0$.

$$E(k) = |c|k \sqrt{1 + \frac{k^2}{4m^2 c^2}} = |c|k \left(1 + \frac{k^2}{8m^2 c^2} + \mathcal{O}(k^3) \right) \approx |c|k + \frac{k^3}{8m^2 |c|}.$$

And as $k \rightarrow \infty$, $k^2 \ll k^4$ and so,

$$E(k) \xrightarrow[k \rightarrow \infty]{} e^{-\beta \epsilon} \rightarrow \frac{k^2}{2|m|}.$$

Therefore, the function is initially linear around $k = 0$ and approaches a parabola as $|k| \gg 0$.

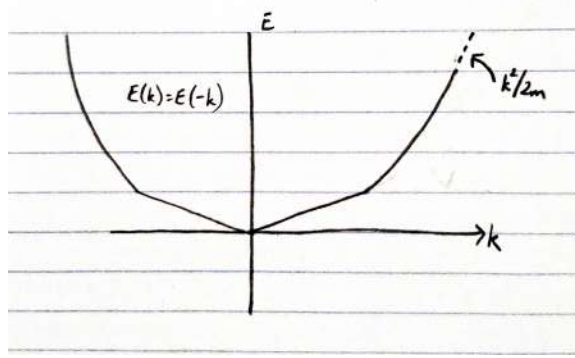


Figure 3: Sketch for Problem 3.1.1

2)

$$\epsilon(k) = \sqrt{\left(\frac{k^2}{2m} - \mu \right)^2 + \Delta^2},$$

where $\mu > 0$, $\Delta > 0$, $m > 0$ and $\mu \gg \Delta$.

Solution: The function is even since $\epsilon(k) = \epsilon(-k)$. There are three “interesting” points: 0, $\sqrt{2m\mu}$, and ∞ . The Taylor series around $k = 0$ is

$$\epsilon(k) \approx \sqrt{\mu^2 + \Delta^2} - \frac{\mu}{2m\sqrt{\mu^2 + \Delta^2}} \cdot k^2,$$

which is a downward-opening parabola. Around $k = \sqrt{2m\mu}$ b

$$\epsilon(k) \approx \Delta - \frac{\mu}{m\Delta} \cdot \left(k - \sqrt{2m\mu} \right)^2,$$

which is an upward-opening parabola. Finally, as $k \rightarrow \infty$, $E(k) \rightarrow k^2/2m$.

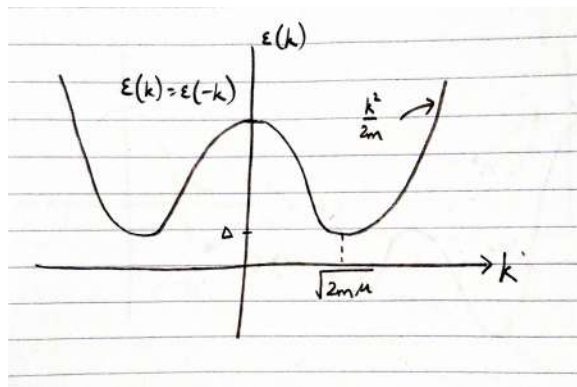


Figure 4: Sketch for Problem 3.1.2

3.2 Taylor Series

1) Find the full series for $(x + 1)^3$ around $x = 0$.

Solution: We first compute the derivatives of $g(x) = (x + 1)^3$:

$$\begin{aligned} g(x) &= (x + 1)^3, \\ g'(x) &= 3(x + 1)^2, \\ g''(x) &= 6(x + 1), \\ g^{(3)}(x) &= 6, \\ g^{(n)}(x) &= 0 \text{ for all } n \geq 4. \end{aligned}$$

Therefore, the Taylor expansion around $x = 0$ is given by

$$\begin{aligned} (x + 1)^3 &= 1 + 3x + \frac{1}{2!} \cdot 6x^2 + \frac{1}{3!} \cdot 6x^3 \\ &= 1 + 3x + 3x^2 + x^3. \end{aligned}$$

The Taylor series of a polynomial terminates after a finite number of terms.

2) Sometimes the expressions on both sides of (1) exist, but the two sides are not equal. Consider the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

Sketch it, then try to compute the Taylor series around $x = 0$.

Solution: At $x = 0$ (that is, when we approach from the left,) all derivatives are zero and so, the Taylor series is $f(x) = 0$. But as we approach from the right, the first derivative

$$f'(0^+) = \lim_{h \rightarrow 0^+} \frac{1/h}{e^{1/h}} = 0.$$

Therefore, all derivatives vanish as we approach from the right as well, and $f(x) = 0$ is indeed the Taylor series. Unlike for $\sqrt{1-x^2}$, the Taylor series is defined in this case, but it does not equal the function.

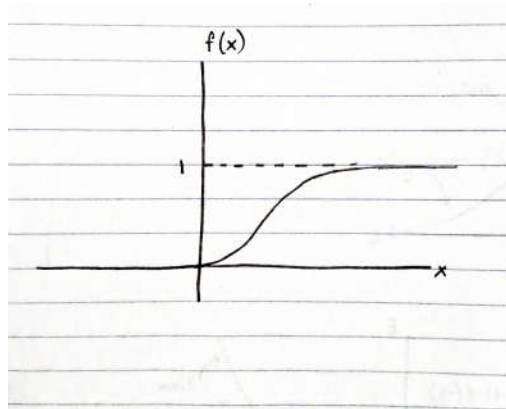


Figure 5: Sketch of the function in Problem 3.2.2