

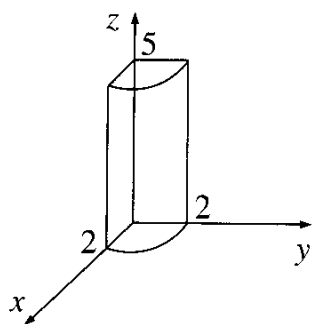


Figure 1.43

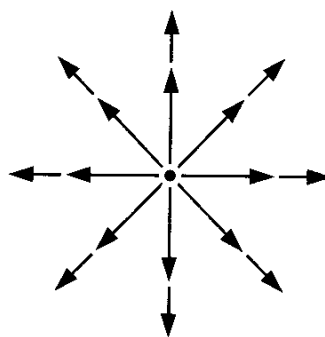


Figure 1.44

Problem 1.42

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2 \phi) \hat{\mathbf{s}} + s \sin \phi \cos \phi \hat{\boldsymbol{\phi}} + 3z \hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

1.5 The Dirac Delta Function

1.5.1 The Divergence of $\hat{\mathbf{r}}/r^2$

Consider the vector function

$$\mathbf{v} = \frac{1}{r^2} \hat{\mathbf{r}}. \quad (1.83)$$

At every location, $\mathbf{v}$ is directed radially outward (Fig. 1.44); if ever there was a function that ought to have a large positive divergence, this is it. And yet, when you actually *calculate* the divergence (using Eq. 1.71), you get precisely *zero*:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0. \quad (1.84)$$

(You will have encountered this paradox already, if you worked Prob. 1.16.) The plot thickens if you apply the divergence theorem to this function. Suppose we integrate over a sphere of radius R , centered at the origin (Prob. 1.38b); the surface integral is

$$\begin{aligned} \oint \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{R^2} \hat{\mathbf{r}} \right) \cdot (R^2 \sin \theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) \\ &= \left(\int_0^\pi \sin \theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) = 4\pi. \end{aligned} \quad (1.85)$$

But the *volume* integral, $\int \nabla \cdot \mathbf{v} \, d\tau$, is *zero*, if we are really to believe Eq. 1.84. Does this mean that the divergence theorem is false? What’s going on here?

The source of the problem is the point $r = 0$, where $\mathbf{v}$ blows up (and where, in Eq. 1.84, we have unwittingly divided by zero). It is quite true that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* the origin, but right *at* the origin the situation is more complicated. Notice that the surface integral (1.85) is *independent of R* ; if the divergence theorem is right (and it *is*), we should get $\int (\nabla \cdot \mathbf{v}) \, d\tau = 4\pi$ for *any* sphere centered at the origin, no matter how small. Evidently the entire contribution must be coming from the point $r = 0$! Thus, $\nabla \cdot \mathbf{v}$ has the bizarre property that it vanishes everywhere except at one point, and yet its *integral* (over any volume containing that point) is 4π . No ordinary function behaves like that. (On the other hand, a *physical* example *does* come to mind: the density (mass per unit volume) of a point particle. It’s zero except at the exact location of the particle, and yet its *integral* is finite—namely, the mass of the particle.) What we have stumbled on is a mathematical object known to physicists as the **Dirac delta function**. It arises in many branches of theoretical physics. Moreover, the specific problem at hand (the divergence of the function $\hat{\mathbf{r}}/r^2$) is not just some arcane curiosity—it is, in fact, central to the whole theory of electrodynamics. So it is worthwhile to pause here and study the Dirac delta function with some care.

1.5.2 The One-Dimensional Dirac Delta Function

The one dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow “spike,” with area 1 (Fig. 1.45). That is to say:

$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\} \tag{1.86}$$

and

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1. \tag{1.87}$$

Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$. In the mathematical literature it is known as a **generalized function**, or **distribution**. It is, if you

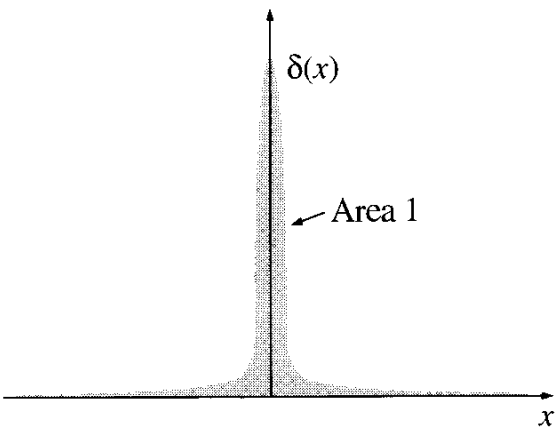


Figure 1.45

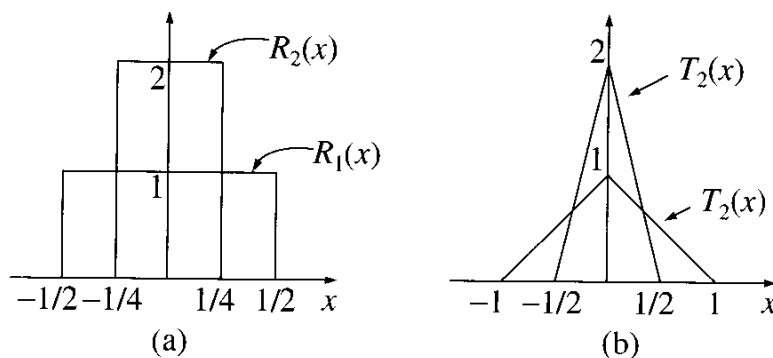


Figure 1.46

like, the *limit* of a *sequence* of functions, such as rectangles $R_n(x)$, of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$ (Fig. 1.46).

If $f(x)$ is some “ordinary” function (that is, *not* another delta function—in fact, just to be on the safe side let’s say that $f(x)$ is *continuous*), then the *product* $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows that

$$f(x)\delta(x) = f(0)\delta(x). \quad (1.88)$$

(This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway *except* at $x = 0$, we may as well replace $f(x)$ by the value it assumes at the origin.) In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0). \quad (1.89)$$

Under an integral, then, the delta function “picks out” the value of $f(x)$ at $x = 0$. (Here and below, the integral need not run from $-\infty$ to $+\infty$; it is sufficient that the domain extend across the delta function, and $-\epsilon$ to $+\epsilon$ would do as well.)

Of course, we can shift the spike from $x = 0$ to some other point, $x = a$ (Fig. 1.47):

$$\delta(x - a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(x - a) dx = 1. \quad (1.90)$$

Equation 1.88 becomes

$$f(x)\delta(x - a) = f(a)\delta(x - a), \quad (1.91)$$

and Eq. 1.89 generalizes to

$$\int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a). \quad (1.92)$$

Example 1.14

Evaluate the integral

$$\int_0^3 x^3 \delta(x - 2) dx.$$

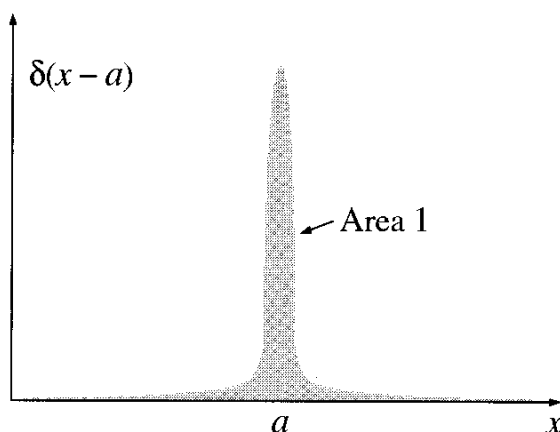


Figure 1.47

Solution: The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3) the answer would be 0, because the spike would then be outside the domain of integration.

Although δ itself is not a legitimate function, *integrals* over δ are perfectly acceptable. In fact, it's best to think of the delta function as something that is *always intended for use under an integral sign*. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if ⁶

$$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx, \quad (1.93)$$

for all ("ordinary") functions $f(x)$.

Example 1.15

Show that

$$\delta(kx) = \frac{1}{|k|} \delta(x), \quad (1.94)$$

where k is any (nonzero) constant. (In particular, $\delta(-x) = \delta(x)$.)

Solution: For an arbitrary test function $f(x)$, consider the integral

$$\int_{-\infty}^{\infty} f(x) \delta(kx) dx.$$

Changing variables, we let $y \equiv kx$, so that $x = y/k$, and $dx = 1/k dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is *negative*, then $x = \infty$ implies $y = -\infty$, and

⁶This is not as arbitrary as it may sound. The crucial point is that the integrals must be equal for *any* $f(x)$. Suppose $D_1(x)$ and $D_2(x)$ actually *differed*, say, in the neighborhood of the point $x = 17$. Then we could pick a function $f(x)$ that was sharply peaked about $x = 17$, and the integrals would not be equal.

vice versa, so the order of the limits is reversed. Restoring the “proper” order costs a minus sign. Thus

$$\int_{-\infty}^{\infty} f(x) \delta(kx) dx = \pm \int_{-\infty}^{\infty} f(y/k) \delta(y) \frac{dy}{k} = \pm \frac{1}{k} f(0) = \frac{1}{|k|} f(0).$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k , as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x) \delta(kx) dx = \int_{-\infty}^{\infty} f(x) \left[\frac{1}{|k|} \delta(x) \right] dx.$$

According to criterion 1.93, therefore, $\delta(kx)$ and $(1/|k|)\delta(x)$ are equal.

Problem 1.43 Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1) \delta(x - 3) dx.$

(b) $\int_0^5 \cos x \delta(x - \pi) dx.$

(c) $\int_0^3 x^3 \delta(x + 1) dx.$

(d) $\int_{-\infty}^{\infty} \ln(x + 3) \delta(x + 2) dx.$

Problem 1.44 Evaluate the following integrals:

(a) $\int_{-2}^2 (2x + 3) \delta(3x) dx.$

(b) $\int_0^2 (x^3 + 3x + 2) \delta(1 - x) dx.$

(c) $\int_{-1}^1 9x^2 \delta(3x + 1) dx.$

(d) $\int_{-\infty}^a \delta(x - b) dx.$

Problem 1.45

(a) Show that

$$x \frac{d}{dx} (\delta(x)) = -\delta(x).$$

[Hint: Use integration by parts.]

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}. \quad (1.95)$$

Show that $d\theta/dx = \delta(x)$.

1.5.3 The Three-Dimensional Delta Function

It is an easy matter to generalize the delta function to three dimensions:

$$\delta^3(\mathbf{r}) = \delta(x) \delta(y) \delta(z). \quad (1.96)$$

(As always, $\mathbf{r} \equiv x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}$ is the position vector, extending from the origin to the point (x, y, z)). This three-dimensional delta function is zero everywhere except at $(0, 0, 0)$, where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\mathbf{r}) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) dx dy dz = 1. \quad (1.97)$$

And, generalizing Eq. 1.92,

$$\int_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{a}) d\tau = f(\mathbf{a}). \quad (1.98)$$

As in the one-dimensional case, integration with δ picks out the value of the function f at the location of the spike.

We are now in a position to resolve the paradox introduced in Sect. 1.5.1. As you will recall, we found that the divergence of $\hat{\mathbf{r}}/r^2$ is zero everywhere except at the origin, and yet its *integral* over any volume containing the origin is a constant (to wit: 4π). These are precisely the defining conditions for the Dirac delta function; evidently

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}). \quad (1.99)$$

More generally,

$$\boxed{\nabla \cdot \left(\frac{\hat{\mathbf{z}}}{z^2} \right) = 4\pi \delta^3(\mathbf{z})}, \quad (1.100)$$

where, as always, $\mathbf{z}$ is the separation vector: $\mathbf{z} \equiv \mathbf{r} - \mathbf{r}'$. Note that differentiation here is with respect to $\mathbf{r}$, while $\mathbf{r}'$ is held constant. Incidentally, since

$$\nabla \cdot \left(\frac{\hat{\mathbf{z}}}{z^2} \right) = -\frac{\hat{\mathbf{z}}}{z^2} \quad (1.101)$$

(Prob. 1.13), it follows that

$$\nabla^2 \frac{1}{z} = -4\pi \delta^3(\mathbf{z}). \quad (1.102)$$

Example 1.16

Evaluate the integral

$$J = \int_{\mathcal{V}} (r^2 + 2) \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) d\tau,$$

where $\mathcal{V}$ is a sphere of radius R centered at the origin.

Solution 1: Use Eq. 1.99 to rewrite the divergence, and Eq. 1.98 to do the integral:

$$J = \int_{\mathcal{V}} (r^2 + 2) 4\pi \delta^3(\mathbf{r}) d\tau = 4\pi(0 + 2) = 8\pi.$$

This one-line solution demonstrates something of the power and beauty of the delta function, but I would like to show you a second method, which is much more cumbersome but serves to illustrate the method of integration by parts, Sect. 1.3.6.

Solution 2: Using Eq. 1.59, we transfer the derivative from $\hat{\mathbf{r}}/r^2$ to $(r^2 + 2)$:

$$J = - \int \frac{\hat{\mathbf{r}}}{r^2} \cdot [\nabla(r^2 + 2)] d\tau + \oint (r^2 + 2) \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}.$$

The gradient is

$$\nabla(r^2 + 2) = 2r\hat{\mathbf{r}},$$

so the volume integral becomes

$$\int \frac{2}{r} d\tau = \int \frac{2}{r} r^2 \sin\theta dr d\theta d\phi = 8\pi \int_0^R r dr = 4\pi R^2.$$

Meanwhile, on the boundary of the sphere (where $r = R$),

$$d\mathbf{a} = R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}},$$

so the surface integral becomes

$$\int (R^2 + 2) \sin\theta d\theta d\phi = 4\pi(R^2 + 2).$$

Putting it all together, then,

$$J = -4\pi R^2 + 4\pi(R^2 + 2) = 8\pi,$$

as before.

Problem 1.46

- (a) Write an expression for the electric charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q .
- (b) What is the charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$?
- (c) What is the charge density of a uniform, infinitesimally thin spherical shell of radius R and total charge Q , centered at the origin? [*Beware:* the integral over all space must equal Q .]

Problem 1.47 Evaluate the following integrals:

- (a) $\int_{\text{all space}} (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2) \delta^3(\mathbf{r} - \mathbf{a}) d\tau$, where $\mathbf{a}$ is a fixed vector and a is its magnitude.
- (b) $\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r}) d\tau$, where $\mathcal{V}$ is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$.
- (c) $\int_{\mathcal{V}} (r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4) \delta^3(\mathbf{r} - \mathbf{c}) d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$, and c is its magnitude.
- (d) $\int_{\mathcal{V}} \mathbf{r} \cdot (\mathbf{d} - \mathbf{r}) \delta^3(\mathbf{e} - \mathbf{r}) d\tau$, where $\mathbf{d} = (1, 2, 3)$, $\mathbf{e} = (3, 2, 1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at $(2, 2, 2)$.

Problem 1.48 Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

(where $\mathcal{V}$ is a sphere of radius R , centered at the origin) by two different methods, as in Ex. 1.16.

1.6 The Theory of Vector Fields

1.6.1 The Helmholtz Theorem

Ever since Faraday, the laws of electricity and magnetism have been expressed in terms of **electric** and **magnetic fields**, $\mathbf{E}$ and $\mathbf{B}$. Like many physical laws, these are most compactly expressed as differential equations. Since $\mathbf{E}$ and $\mathbf{B}$ are *vectors*, the differential equations naturally involve vector derivatives: divergence and curl. Indeed, Maxwell reduced the entire theory to four equations, specifying respectively the divergence and the curl of $\mathbf{E}$ and $\mathbf{B}$.⁷

⁷Strictly speaking, this is only true in the static case; in general, the divergence and curl are given in terms of time derivatives of the fields themselves.