

Here again the integrand is the product of one function (f) and the derivative (in this case the *divergence*) of another ($\mathbf{A}$), and integration by parts licenses us to transfer the derivative from $\mathbf{A}$ to f (where it becomes a *gradient*), at the cost of a minus sign and a boundary term (in this case a surface integral).

You might wonder how often one is likely to encounter an integral involving the product of one function and the derivative of another; the answer is *surprisingly* often, and integration by parts turns out to be one of the most powerful tools in vector calculus.

Problem 1.35

(a) Show that

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_S [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_P f \mathbf{A} \cdot d\mathbf{l}. \quad (1.60)$$

(b) Show that

$$\int_V \mathbf{B} \cdot (\nabla \times \mathbf{A}) d\tau = \int_V \mathbf{A} \cdot (\nabla \times \mathbf{B}) d\tau + \oint_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}. \quad (1.61)$$

1.4 Curvilinear Coordinates

1.4.1 Spherical Polar Coordinates

The spherical polar coordinates (r, θ, ϕ) of a point P are defined in Fig. 1.36; r is the distance from the origin (the magnitude of the position vector), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates (x, y, z) can be read from the figure:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (1.62)$$

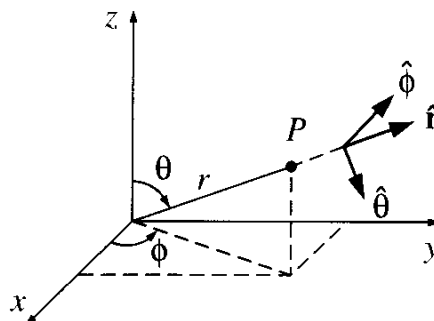


Figure 1.36

Figure 1.36 also shows three unit vectors, $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$), and any vector $\mathbf{A}$ can be expressed in terms of them in the usual way:

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}. \quad (1.63)$$

A_r , A_θ , and A_ϕ are the radial, polar, and azimuthal components of $\mathbf{A}$. In terms of the Cartesian unit vectors,

$$\left. \begin{aligned} \hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}, \end{aligned} \right\} \quad (1.64)$$

as you can easily check for yourself (Prob. 1.37). I have put these formulas inside the back cover, for easy reference.

But there is a poisonous snake lurking here that I'd better warn you about: $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}},$ and $\hat{\boldsymbol{\phi}}$ are associated with a *particular point* P , and they *change direction* as P moves around. For example, $\hat{\mathbf{r}}$ always points radially outward, but “radially outward” can be the x direction, the y direction, or any other direction, depending on where you are. In Fig. 1.37, $\mathbf{A} = \hat{\mathbf{y}}$ and $\mathbf{B} = -\hat{\mathbf{y}}$, and yet *both* of them would be written as $\hat{\mathbf{r}}$ in spherical coordinates. One could take account of this by explicitly indicating the point of reference: $\hat{\mathbf{r}}(\theta, \phi), \hat{\boldsymbol{\theta}}(\theta, \phi), \hat{\boldsymbol{\phi}}(\theta, \phi)$, but this would be cumbersome, and as long as you are alert to the problem I don't think it will cause difficulties.⁴ In particular, do not naïvely combine the spherical components of vectors associated with different points (in Fig. 1.37, $\mathbf{A} + \mathbf{B} = 0$, not $2\hat{\mathbf{r}}$, and $\mathbf{A} \cdot \mathbf{B} = -1$, not $+1$). Beware of differentiating a vector that is expressed in spherical coordinates, since the unit vectors themselves are functions of position ($\partial \hat{\mathbf{r}} / \partial \theta = \hat{\boldsymbol{\theta}}$, for example). And do not take $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}},$ and $\hat{\boldsymbol{\phi}}$ outside an integral, as we did with $\hat{\mathbf{x}}, \hat{\mathbf{y}},$ and $\hat{\mathbf{z}}$ in Eq. 1.53. In general, if you're uncertain about the validity of an operation, reexpress the problem in Cartesian coordinates, where this difficulty does not arise.

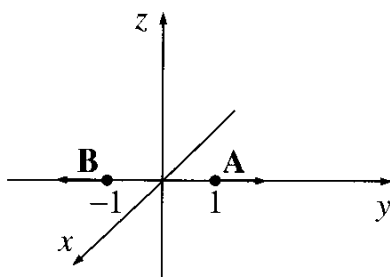


Figure 1.37

⁴I claimed on the very first page that vectors have no location, and I'll stand by that. The vectors themselves live “out there,” completely independent of our choice of coordinates. But the *notation* we use to represent them *does* depend on the point in question, in curvilinear coordinates.

An infinitesimal displacement in the $\hat{\mathbf{r}}$ direction is simply dr (Fig. 1.38a), just as an infinitesimal element of length in the x direction is dx :

$$dl_r = dr. \quad (1.65)$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (Fig. 1.38b) is *not* just $d\theta$ (that's an *angle*—it doesn't even have the right *units* for a length), but rather $r d\theta$:

$$dl_\theta = r d\theta. \quad (1.66)$$

Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (Fig. 1.38c) is $r \sin \theta d\phi$:

$$dl_\phi = r \sin \theta d\phi. \quad (1.67)$$

Thus, the general infinitesimal displacement $d\mathbf{l}$ is

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}. \quad (1.68)$$

This plays the role (in line integrals, for example) that $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$ played in Cartesian coordinates.

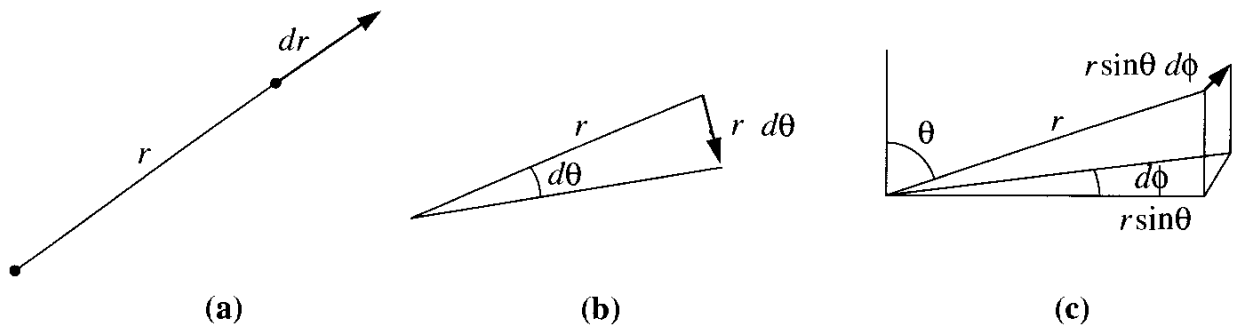


Figure 1.38

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin \theta dr d\theta d\phi. \quad (1.69)$$

I cannot give you a general expression for *surface* elements $d\mathbf{a}$, since these depend on the orientation of the surface. You simply have to analyze the geometry for any given case (this goes for Cartesian and curvilinear coordinates alike). If you are integrating over the surface of a sphere, for instance, then r is constant, whereas θ and ϕ change (Fig. 1.39), so

$$d\mathbf{a}_1 = dl_\theta dl_\phi \hat{\mathbf{r}} = r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}.$$

On the other hand, if the surface lies in the xy plane, say, so that θ is constant (to wit: $\pi/2$) while r and ϕ vary, then

$$d\mathbf{a}_2 = dl_r dl_\phi \hat{\theta} = r dr d\phi \hat{\theta}.$$

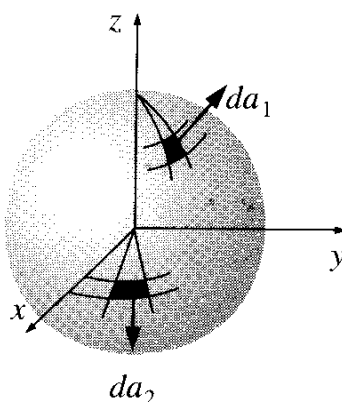


Figure 1.39

Notice, finally, that r ranges from 0 to ∞ , ϕ from 0 to 2π , and θ from 0 to π (*not* 2π —that would count every point twice).⁵

Example 1.13

Find the volume of a sphere of radius R .

Solution:

$$\begin{aligned}
 V &= \int d\tau = \int_{r=0}^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin \theta \, dr \, d\theta \, d\phi \\
 &= \left(\int_0^R r^2 \, dr \right) \left(\int_0^{\pi} \sin \theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \\
 &= \left(\frac{R^3}{3} \right) (2)(2\pi) = \frac{4}{3}\pi R^3.
 \end{aligned}$$

(Not a big surprise.)

So far we have talked only about the *geometry* of spherical coordinates. Now I would like to “translate” the vector derivatives (gradient, divergence, curl, and Laplacian) into r , θ , ϕ notation. In principle this is entirely straightforward: in the case of the gradient,

$$\nabla T = \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}},$$

for instance, we would first use the chain rule to reexpress the partials:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r} \left(\frac{\partial r}{\partial x} \right) + \frac{\partial T}{\partial \theta} \left(\frac{\partial \theta}{\partial x} \right) + \frac{\partial T}{\partial \phi} \left(\frac{\partial \phi}{\partial x} \right).$$

⁵Alternatively, you could run ϕ from 0 to π (the “eastern hemisphere”) and cover the “western hemisphere” by extending θ from π up to 2π . But this is very bad notation, since, among other things, $\sin \theta$ will then run negative, and you’ll have to put absolute value signs around that term in volume and surface elements (area and volume being intrinsically positive quantities).

The terms in parentheses could be worked out from Eq. 1.62—or rather, the *inverse* of those equations (Prob. 1.36). Then we'd do the same for $\partial T/\partial y$ and $\partial T/\partial z$. Finally, we'd substitute in the formulas for $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ (Prob. 1.37). It would take an hour to figure out the gradient in spherical coordinates by this brute-force method. I suppose this is how it was first done, but there is a much more efficient indirect approach, explained in Appendix A, which has the extra advantage of treating all coordinate systems at once. I described the “straightforward” method only to show you that there is nothing subtle or mysterious about transforming to spherical coordinates: you're expressing the *same quantity* (gradient, divergence, or whatever) in different notation, that's all.

Here, then, are the vector derivatives in spherical coordinates:

Gradient:

$$\nabla T = \frac{\partial T}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}}. \quad (1.70)$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}. \quad (1.71)$$

Curl:

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}. \end{aligned} \quad (1.72)$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2}. \quad (1.73)$$

For reference, these formulas are listed inside the front cover.

Problem 1.36 Find formulas for r, θ, ϕ in terms of x, y, z (the inverse, in other words, of Eq. 1.62).

- **Problem 1.37** Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}$, ...). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

- **Problem 1.38**

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2 \hat{\mathbf{r}}$, using as your volume the sphere of radius R , centered at the origin.

b) Do the same for $\mathbf{v}_2 = (1/r^2) \hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

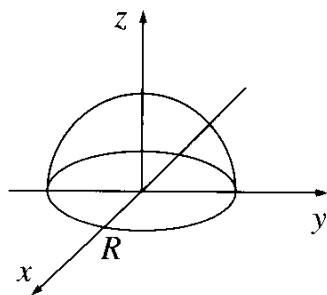


Figure 1.40

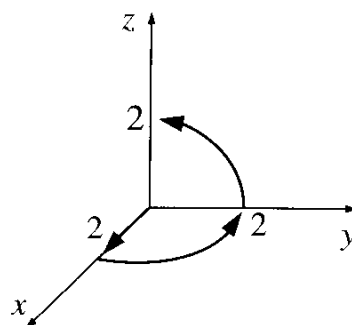


Figure 1.41

Problem 1.39 Compute the divergence of the function

$$\mathbf{v} = (r \cos \theta) \hat{\mathbf{r}} + (r \sin \theta) \hat{\boldsymbol{\theta}} + (r \sin \theta \cos \phi) \hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R , resting on the xy plane and centered at the origin (Fig. 1.40).

Problem 1.40 Compute the gradient and Laplacian of the function $T = r(\cos \theta + \sin \theta \cos \phi)$. Check the Laplacian by converting T to Cartesian coordinates and using Eq. 1.42. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from $(0, 0, 0)$ to $(0, 0, 2)$.

1.4.2 Cylindrical Coordinates

The cylindrical coordinates (s, ϕ, z) of a point P are defined in Fig. 1.42. Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is

$$x = s \cos \phi, \quad y = s \sin \phi, \quad z = z. \quad (1.74)$$

The unit vectors (Prob. 1.41) are

$$\left. \begin{aligned} \hat{\mathbf{s}} &= \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned} \right\} \quad (1.75)$$

The infinitesimal displacements are

$$dl_s = ds, \quad dl_\phi = s d\phi, \quad dl_z = dz, \quad (1.76)$$

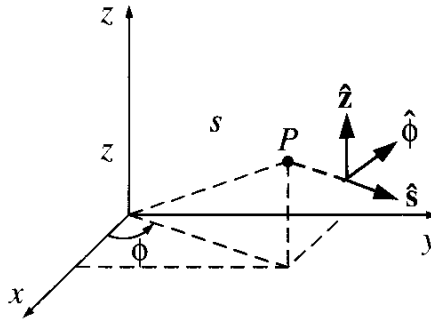


Figure 1.42

so

$$d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}, \quad (1.77)$$

and the volume element is

$$d\tau = s ds d\phi dz. \quad (1.78)$$

The range of s is $0 \rightarrow \infty$, ϕ goes from $0 \rightarrow 2\pi$, and z from $-\infty$ to ∞ .

The vector derivatives in cylindrical coordinates are:

Gradient:

$$\nabla T = \frac{\partial T}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}}. \quad (1.79)$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s}(s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \quad (1.80)$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}. \quad (1.81)$$

Laplacian:

$$\nabla^2 T = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial T}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \quad (1.82)$$

These formulas are also listed inside the front cover.

Problem 1.41 Express the cylindrical unit vectors $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.75). “Invert” your formulas to get $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ (and ϕ).

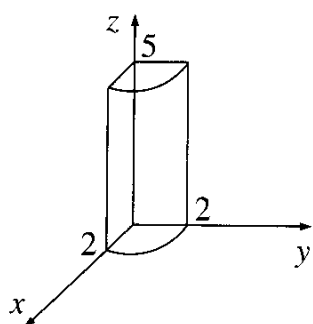


Figure 1.43

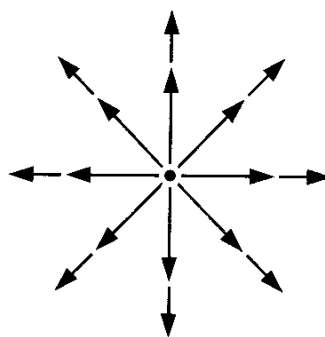


Figure 1.44

Problem 1.42

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2 \phi) \hat{\mathbf{s}} + s \sin \phi \cos \phi \hat{\boldsymbol{\phi}} + 3z \hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

1.5 The Dirac Delta Function

1.5.1 The Divergence of $\hat{\mathbf{r}}/r^2$

Consider the vector function

$$\mathbf{v} = \frac{1}{r^2} \hat{\mathbf{r}}. \quad (1.83)$$

At every location, $\mathbf{v}$ is directed radially outward (Fig. 1.44); if ever there was a function that ought to have a large positive divergence, this is it. And yet, when you actually *calculate* the divergence (using Eq. 1.71), you get precisely *zero*:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0. \quad (1.84)$$

(You will have encountered this paradox already, if you worked Prob. 1.16.) The plot thickens if you apply the divergence theorem to this function. Suppose we integrate over a sphere of radius R , centered at the origin (Prob. 1.38b); the surface integral is

$$\begin{aligned} \oint \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{R^2} \hat{\mathbf{r}} \right) \cdot (R^2 \sin \theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) \\ &= \left(\int_0^\pi \sin \theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) = 4\pi. \end{aligned} \quad (1.85)$$