## B.4 Problem Set 4: Quantum Initial Conditions

### 1. Slow-Roll Inflation

(a) Consider slow-roll inflation with a polynomial potential $V(\phi) = \mu^{4-p}\phi^p$, where $p > 0$ and $\mu$ is a parameter with the dimension of mass. Show that the spectral index $n_s$ and the tensor-to-scalar ratio $r$, evaluated at a reference scale $k_*$ corresponding to CMB fluctuations, are

$$n_s - 1 = -\frac{2+p}{2N_*}\,, \qquad r = \frac{4p}{N_*}\,,$$

where $N_* \approx 50$ is the number of $e$-folds between the horizon exit of $k_*$ and the end of inflation. Which values of $p$ are still consistent with current observations?

(b) Axions are promising inflaton candidates. At the perturbative level, an axion enjoys a continuous shift symmetry, but this is broken nonperturbatively to a discrete symmetry, leading to a potential of the form $V(\phi) = \mu^4[1 - \cos(\phi/f)]$, where $f$ is the axion decay constant. Using this as the inflationary potential, show that

$$n_s - 1 = -\alpha\,\frac{e^{N_*\alpha}+1}{e^{N_*\alpha}-1}\,, \qquad r = 8\alpha\,\frac{1}{e^{N_*\alpha}-1}\,,$$

where $\alpha \equiv M_{\rm pl}^2/f^2$. Sketch this prediction in the $n_s$-$r$ plane. Discuss the limit $\alpha \gg 1$.

### 2. The Lyth Bound

(a) Show that the tensor-to-scalar ratio predicted by slow-roll inflation is

$$r \equiv \frac{A_t}{A_s} = \frac{8\dot{\phi}^2}{M_{\rm pl}^2 H^2}\,.$$

(b) Show that the inflaton field travels a "distance" $\Delta\phi \equiv |\phi_E - \phi_*|$ during (observable) inflation

$$\frac{\Delta\phi}{M_{\rm pl}} = \frac{\Delta N}{60}\sqrt{\frac{r}{0.002}}\,,$$

where $\Delta N$ is the total number of $e$-folds between the time when the CMB scales exited the horizon and the end of inflation. [You may assume that $\varepsilon \approx$ *const.* during inflation] Comment on the implication of this result for observable gravitational waves. [Realistically, we require $r > 0.001$ to have a fighting chance of detecting gravitational waves via CMB polarisation.]

(c) Derive the following relationship between the energy scale of inflation, $V^{1/4}$, and the tensor-to-scalar ratio,

$$V^{1/4} = \left(\frac{3\pi^2}{2}\,rA_s\right)^{1/4} M_{\rm pl}\,.$$

Use $A_s = 2.1 \times 10^{-9}$ to determine $V^{1/4}$ for $r = 0.01$. How does that compare to the energy scales probed by the LHC?