Problem Set #6 Due: Thursday, April 13, 9:15 am

1 Metric perturbations on super-horizon scales

In lecture, we introduced the comoving curvature perturbation

$$\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\bar{\rho} + \bar{P}}\delta q, \quad \text{with} \quad \delta T^0_j \equiv -\partial_j \delta q$$

and proved that it doesn't evolve on superhorizon scales. Use the conservation of $\mathcal{R}$ to show that the gravitational potential decreases by a factor of 9/10 on superhorizon scales in the transition from radiation-dominated to matter-dominated.

2 Growth of matter perturbations

At early times, the universe was dominated by radiation (r) and pressureless matter (m). You may ignore baryons.

(a) Show that the conformal Hubble parameter (i.e. $\mathcal{H}$, not H) satisfies

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r}\left(\frac{1}{y} + \frac{1}{y^2}\right) ,$$

where $y \equiv a/a_{eq}$ is the ratio of the scale factor to its value when the energy density of the matter and radiation are equal.

(b) For perturbations on scales much smaller than the Hubble radius, the fluctuations in the radiation can be neglected. Assuming that Φ evolves on a Hubble timescale, show that

$$\delta_m^{''} + \mathcal{H}\delta_m^{'} - 4\pi G a^2 \bar{\rho}_m \delta_m \approx 0 . \qquad (1)$$

[You may use any equations given in the lectures without proof.]

(c) Show that, in terms of the variable y, eq. (1) becomes

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0 .$$

Verify that the two independent solutions are

$$\delta_m^{(1)} \propto 2 + 3y ,$$

$$\delta_m^{(2)} \propto (2+3y)\log\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y} .$$

By taking the appropriate limits determine how δ_m grows with y for $y \ll 1$ (RD) and $y \gg 1$ (MD).

3 Cosmological gravitational waves

Tensor perturbations of the metric which survive until today from the early universe are called cosmological gravitational waves. (The gravitational waves recently discovered by LIGO are also tensor perturbations of the metric; however they are are generated "today" by violent astrophysical events such as BH-BH inspirals.)

The line element of a FRW metric with tensor perturbations is

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + 2h_{ij})dx^i dx^j\right] ,$$

where h_{ij} is symmetric, traceless and transverse. To linear order in h_{ij}, the non-zero connection coefficients are

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} , \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + 2\mathcal{H}h_{ij} + h'_{ij} , \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j + \delta^{il}h'_{lj} , \\
\Gamma^i_{jk} &= \partial_j h^i_k + \partial_k h^i_j - \delta^{il}\partial_l h_{jk} .
\end{aligned}$$

(a) Show that the perturbation to the Einstein tensor has non-zero components

$$\delta G_{ij} = h''_{ij} - \nabla^2 h_{ij} + 2\mathcal{H}h'_{ij} - 2h_{ij}(2\mathcal{H}' + \mathcal{H}^2) .$$

[Hint: Convince yourself that the Ricci scalar has no tensor perturbations at first order.]

(b) Combine the previous result with the perturbation to the stress tensor, $\delta T_{ij} = 2a^2\bar{P}h_{ij} - a^2\Pi_{ij}$, to show that the perturbed Einstein equation reduces to

$$h''_{ij} - \nabla^2 h_{ij} + 2\mathcal{H}h'_{ij} = -8\pi G a^2 \Pi_{ij} .$$

(c) For the case where $\nabla^2 h_{ij} = -k^2 h_{ij}$ (i.e. a Fourier mode of the metric perturbation), and assuming the anisotropic stress can be ignored, show that

$$h_{ij} \propto \frac{k\eta\cos(k\eta) - \sin(k\eta)}{(k\eta)^3}$$

is a solution for a matter-dominated universe ($a \propto \eta^2$).

(d) Show that the solution tends to a constant for $k\eta \ll 1$ and argue that such a constant solution always exists for super-Hubble gravitational waves irrespective of the equation of state of the matter. For the specifc solution above, show that well inside the Hubble radius it oscillates at (comoving) frequency k and with an amplitude that falls as $1/a$. (This behavior is also general and follows from a WKB solution of the Einstein equation.)