PY - Cosmology Spring 2017

Problem Set #4 Due: Thursday, March 16, 9:15 am

1 Fun with Baumann: thermal history

For this problem you are asked to solve exercises from Baumann's lecture notes. See his notes for the context in which the problems arise.

1. *Relic photons:* Using that the temperature of the CMB today is $T_0 = 2.75K$, show that
$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3 \approx 410 \text{ photons cm}^{-3},$$
$$\rho_{\gamma,0} = \frac{\pi^2}{15} T_0^4 \approx 4.7 \times 10^{-34} g\ cm^{-3} \quad \to \quad \Omega_\gamma h^2 \approx 2.5 \times 10^{-5}.$$
You may use the formulas derived in lecture for number density and energy density in relativistic particles. This is an exercise in unit conversions.

2. *Free streaming particles:* This relates to the CMB photons of the previous question (and also to neutrinos after neutrino decoupling). During the production of the CMB photons are produced in thermal equilibrium, however subsequently they free-stream through the expanding universe without collisions and are therefore not in thermal equilibrium. Nonetheless the photon phase space distribution function retains it's equilibrium shape (with an appropriate redshift of the photon temperature). Derive this result using the microscopic picture that photons are neither created or destroyed during free-streaming and using what you know about the redshifting of momenta in the expanding universe. Use your result for the redshifting of the phase space distribution function to verify the expressions for the number density and energy density of the CMB photons today (which you used in the previous problem without questioning their validity).

3. *Non-relativistic energy and pressure:* Derive formulas (3.2.50) and (3.2.51) in Baumann.

4. *Pressure in equilibrium:* Derive formula (3.2.58) in Baumann. Hint: Note that the phase space distribution is a function of E/T.

2 Chemical potential for electrons

1. Show that the difference between the number densities of electrons and positrons in the relativistic limit ($m_e \ll T$) is
$$n_e - n_{\bar{e}} = \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right] ,$$

where μ_e is the chemical potential of electrons. Note that this formula is exact in the limit of vanishing electron mass, i.e. it is not an expansion in μ_e/T. Hint: you may find the formula $\int_0^\infty dy\, y/(e^y+1) = \pi^2/12$ useful.

2. Electrical neutrality of the universe implies that the number of protons n_p is equal to $n_e - n_{\bar{e}}$ today (there are no anti-protons). Charge conservation then implies that $n_p = n_e - n_{\bar{e}}$ also in the early universe at temperatures for which electrons are relativistic while protons are not. Use this and the formula you derived in the previous problem to estimate μ_e/T (for $m_p \gg T \gg m_e$). You may use that the number of protons today is approximately equal to the number of baryons and that the baryon energy density satisfies $\Omega_b h^2 \sim 0.02$. Don't worry about factors of order one, just find the order of magnitude.

3 Massive cosmological neutrinos

Assume that only one of the three neutrino species has a non-negligible mass m_ν which is much smaller than the neutrino decoupling temperature $T_{dec} \sim 1$ MeV, so that all neutrinos are relativistic when they decouple. Compute the temperature of the neutrinos relative to the cosmic microwave photons after neutrino decoupling and use this to estimate their number density. Show that the energy density that these neutrinos contribute to the universe today is

$$\Omega_\nu h^2 \approx \frac{m_\nu}{94\,\mathrm{eV}} \,.$$