1. De Sitter Space

(a) Show in the context of expanding FRW models that if the combination $\rho + 3P$ is always positive, then there was a Big Bang singularity in the past. [A sketch of $a(t)$ vs. t may be helpful.]

(b) Show that the line element for a positively curved FRW model ($k = +1$) with only vacuum energy ($P = -\rho$) is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \ell^2 \cosh^2(t/\ell)\left[\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2\right].$$

Does this model have an initial Big Bang singularity?

2. Friedmann Equation

Consider a universe with pressureless matter, a cosmological constant and spatial curvature.

(a) Show that the Friedmann equation can be written as the equation of motion of a particle moving in one dimension with total energy zero and potential

$$V(a) = -\frac{4\pi G}{3}\frac{\rho_{m,0}}{a} + \frac{k}{2} - \frac{\Lambda}{6}a^2 \,,$$

where $\Lambda \equiv 8\pi G\rho_\Lambda = const$, $\rho_{m,0} \equiv \rho_m(t_0)$ and $a_0 \equiv a(t_0) \equiv 1$. Sketch $V(a)$ for the following cases: *i)* $k = 0$, $\Lambda < 0$, *ii)* $k \neq 0$, $\Lambda = 0$, and *iii)* $k = 0$, $\Lambda > 0$. Assuming that the universe "starts" with $da/dt > 0$ near $a = 0$, describe the evolution in each case. Where applicable determine the maximal value of the scale factor.

(b) Now consider the case $k > 0$, $\Lambda = 0$. Show that the normalization of the scale factor, $a_0 \equiv 1$, implies $k = H_0^2(\Omega_{m,0} - 1)$. Rewrite the Friedmann equation in conformal time and confirm that the following is a solution

$$a(\eta) = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}\left[1 - \cos(\sqrt{k}\eta)\right].$$

Integrate this result to obtain

$$t(\eta) = H_0^{-1}\frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)^{3/2}}\left[\sqrt{k}\eta - \sin(\sqrt{k}\eta)\right].$$

Show that the universe collapses to a 'Big Crunch' at $t_{\mathrm{BC}} = \pi H_0^{-1}\Omega_{m,0}(\Omega_{m,0} - 1)^{-3/2}$. How many times can a photon circle this universe before t_{BC}?

3. Flatness Problem

Consider an FRW model dominated by a perfect fluid with pressure $P = w\rho$, for $w = const.$ Let the time-dependent density parameter be

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\rm crit}(t)}\,,$$

where $\rho_{\rm crit}(t) \equiv 3H^2/8\pi G$. Show that

$$\frac{d\Omega}{d\ln a} = (1+3w)\,\Omega(\Omega - 1)\,.$$

Discuss the evolution of $\Omega(a)$ for different values of w.

4. Einstein's Biggest Blunder

(a) Consider a universe filled with a perfect fluid with $\rho > 0$ and $P \geq 0$. Show there is no static isotropic homogeneous solution to Einstein's equations.

(b) Now, consider a universe filled with pressureless matter ($P_m = 0$) and allow for a cosmological constant Λ in the Einstein equation, $G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$. Show that it is possible to obtain a static solution if

$$\Lambda = 4\pi G \rho_{m,0} \,.$$

However, show that this solution is unstable to small perturbations $\delta\rho_m \ll \rho_m$.

5. Accelerating Universe

Consider flat FRW models ($k = 0$) with pressureless matter ($P_m = 0$) and a non-zero cosmological constant $\Lambda \neq 0$, that is, with $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$.

(a) Show that the normalised solution ($a_0 \equiv 1$) for $\Omega_{m,0} \neq 0$ can be written as

$$a(t) = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3} \left(\sinh\left[\tfrac{3}{2} H_0 (1 - \Omega_{m,0})^{1/2} t\right]\right)^{2/3} .$$

Verify that $a(t)$ has the expected limits at early times, $H_0 t \ll 1$, and at late times, $H_0 t \gg 1$. Hence show that the age of the universe t_0 in these models is

$$t_0 = \tfrac{2}{3} H_0^{-1} (1 - \Omega_{m,0})^{-1/2} \sinh^{-1}\left[(1/\Omega_{m,0} - 1)^{1/2}\right] ,$$

and roughly sketch this as a function of $\Omega_{m,0}$.

(b) Show that the energy density of the universe becomes dominated by the cosmological constant term at the following redshift

$$1 + z_\Lambda = \left(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3} ,$$

but that it begins accelerating earlier at $1 + z_{\mathrm{a}} = 2^{\frac{1}{3}} (1 + z_\Lambda)$.