working toward grand unification

need some group theory, SU(N) group and representations

example SU(3) (QCD):

special unitary transformations U: $q^i \to U^i{}_j q^j$ $i = 1 \ldots 3$
$1 \ldots N$

$\uparrow$ special: $\det U = 1$ $\uparrow$ unitary: $U^\dagger U = \mathbb{1}$

can write $U = e^{i\alpha^a T^a}$, T^a hermitian, traceless $a = 1 \ldots 8$

$T^\dagger = T$ $\mathrm{Tr}\, T = 0$

Gell-Mann matrices: $T^{1,2,3} = \frac{1}{2}\begin{pmatrix} \sigma^a & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$T^{4,5} = \frac{1}{2}\begin{pmatrix} & & 1, -i \\ & & \\ 1, i & & \end{pmatrix}$

$T^{6,7} = \frac{1}{2}\begin{pmatrix} & & \\ & & 1, -i \\ & 1, i & \end{pmatrix}$

$T^{8} = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

Normalization: $\mathrm{Tr}(T^a T^b) \equiv (T^a)^i{}_j (T^b)^j{}_i = \frac{1}{2}\delta^{ab}$

same for $su(2)$ $\quad \mathrm{Tr}\left(\frac{1}{2}\sigma^a \frac{1}{2}\sigma^b\right) = \frac{1}{2}\delta^{ab}$

also for $su(N)$.

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{\text{quarks}} q_L^\dagger i\bar{\sigma}^\mu D_\mu q_L + \underbrace{q_R^\dagger i\sigma^\mu D_\mu q_R}_{\equiv\, q^{c\dagger} i\bar{\sigma}^\mu D_\mu q^c}$$

+ quark mass terms

$$D_\mu q = (\partial_\mu - i g_s A^a_\mu T^a + su(2), u(1)_Y)\, q$$

A^a_μ: 8 gluon fields

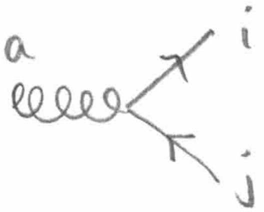
note $q \to e^{i\alpha T} q \Rightarrow q^c \to e^{-i\alpha T} q^c$

$$\Rightarrow D_\mu q^c = (\partial_\mu + i g_s A^a_\mu T^a)\, q^c$$

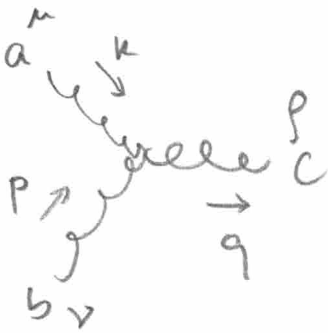
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$

$$[T^a, T^b] \equiv i f^{abc} T^c \quad \text{structure constants}$$

Feynman rules:



$$-ig_s T^{a\,i}{}_{j}$$

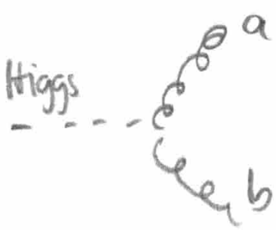


$$\sim g_s f^{abc} \left(g^{\mu\nu} q^{\rho} + \text{permutations} \right)$$

$$\sim g_s^2 f f$$

color factors (sum & average):

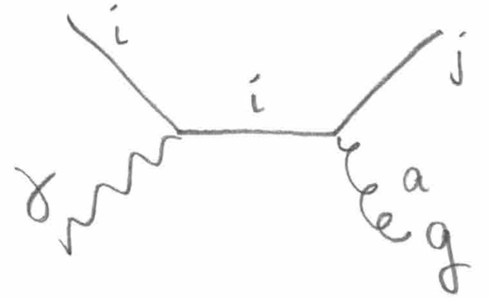
Ex1.



$$M \propto \delta^{ab}$$

$$\Gamma \propto \sum_{a,b} |M|^2 = \sum_{a,b} \delta^{ab} \delta^{ba} = \sum_{a} \delta^{aa} = 8$$

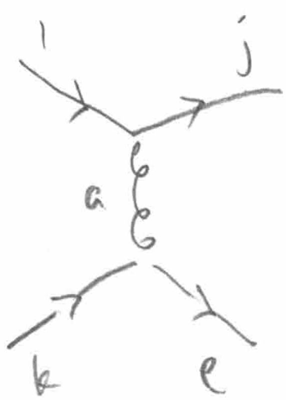
$$\frac{1}{8 \cdot 8} \sum |M|^2 = \frac{1}{8}$$



$\sim T^{a\,i}{}_{j}$

$$\frac{1}{3}\sum_{aij}\left|T^{a\,i}{}_{j}\right|^2 = \frac{1}{3}\sum T^{a\,i}{}_{j}\underbrace{\left(T^{a}{}_{i}{}^{j}\right)^*}_{\left(T^{a\dagger}\right)^{j}{}_{i} = T^{a\,j}{}_{i}}$$

$$= \frac{1}{3}\sum_a \mathrm{tr}(T^aT^a) = \frac{1}{3}\,\frac{1}{2}\sum_a \delta^{aa} = \frac{8}{6} = \frac{4}{3}$$

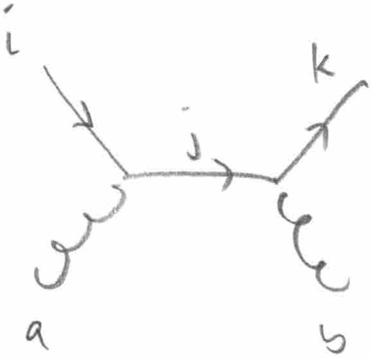


already summed

$\sim T^{a\,i}{}_{j}\,T^{a\,k}{}_{e}$

$$\frac{1}{9}\sum_{\substack{aijke\\b}} \underbrace{\underbrace{T^{a\,i}{}_{j}\,T^{b\,j}{}_{i}}_{\frac{1}{2}\delta^{ab}}\;\underbrace{T^{a\,k}{}_{e}\,T^{b\,e}{}_{k}}_{\frac{1}{2}\delta^{ab}}}_{\frac{1}{4}\delta^{aa} = 2}$$

$\Rightarrow\ 2/9$



$$\sim (T^a T^b)^i{}_k$$

$$\frac{1}{3 \cdot 8} \sum_{ab} \mathrm{tr} \underbrace{(T^a T^b T^b T^a)}_{= \mathrm{Tr}\, T^a T^a T^b T^b}$$

use $\sum_a T^{a\,i}{}_j T^{a\,k}{}_e = \frac{1}{2}\left(\delta^i{}_e \delta^k{}_j - \frac{1}{N}\delta^i{}_j \delta^k{}_e\right)$

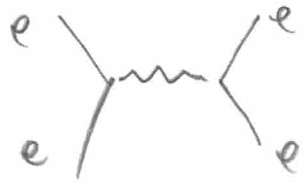
$$\Rightarrow \sum_a (T^a T^a)^i{}_e = \frac{1}{2}\left(3\delta^i{}_e - \frac{1}{3}\delta^i{}_e\right) = \frac{4}{3}\delta^i{}_e$$

$$\Rightarrow \frac{1}{3 \cdot 8} \frac{4}{3} \underbrace{\mathrm{Tr}(T^b T^b)}_{4} = 2/9$$

$\Rightarrow$ Feynman diagrams analogous to QED except

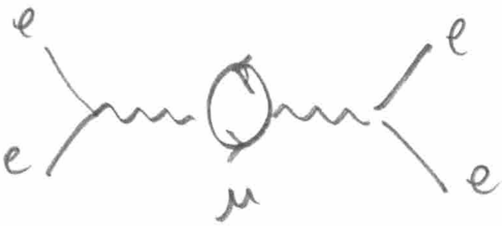
- $\alpha_s(M_Z) = \dfrac{g_s^2(M_Z)}{4\pi} \approx 0.118$
- color factors
- gluon self coupling

Running coupling:

Example QED: 

$M \sim \frac{e^2}{s} (\bar{u}\gamma^\mu v \; \bar{v}\gamma_\mu u)$

photon propagator

higher order 

$M \sim e^4$ (spinors)

$M_{combined} \sim e^2$ spinors (+)

higher order photon propagator

keep going + + + ...

= (1 + + $(\;)^2$ + $(\;)^3$ + ...)

= $\frac{1}{1 - \;}$

What is 

$e^2 \int \frac{d^4\ell}{(2\pi)^4} \, \mathrm{tr}\left[\frac{\not\ell}{\ell^2}\gamma^\mu \frac{\not p + \not\ell}{(p+\ell)^2}\gamma^\nu\right] \frac{1}{p^2}$

↖ ignoring m_μ

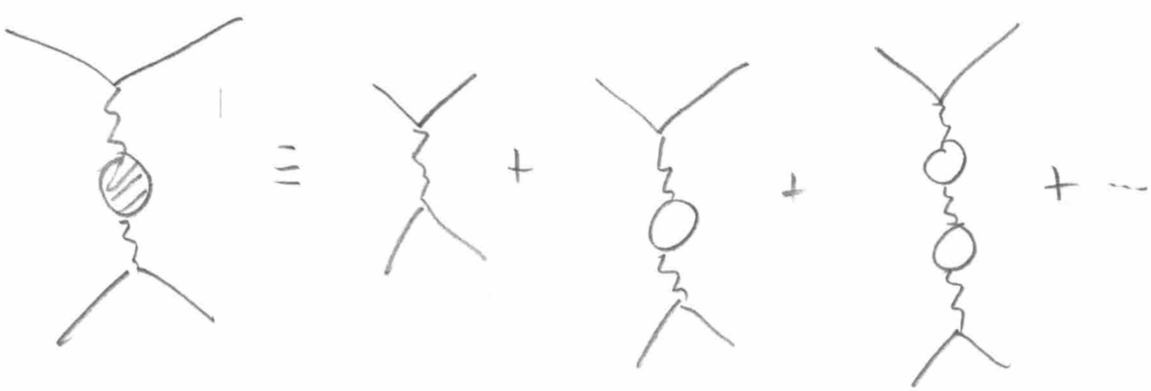
$$= \text{divergent piece} + \frac{\alpha}{3\pi}\log\left(\frac{p^2}{4m_\mu^2}\right)$$

(and zero when $p^2 < 4m_\mu^2$)

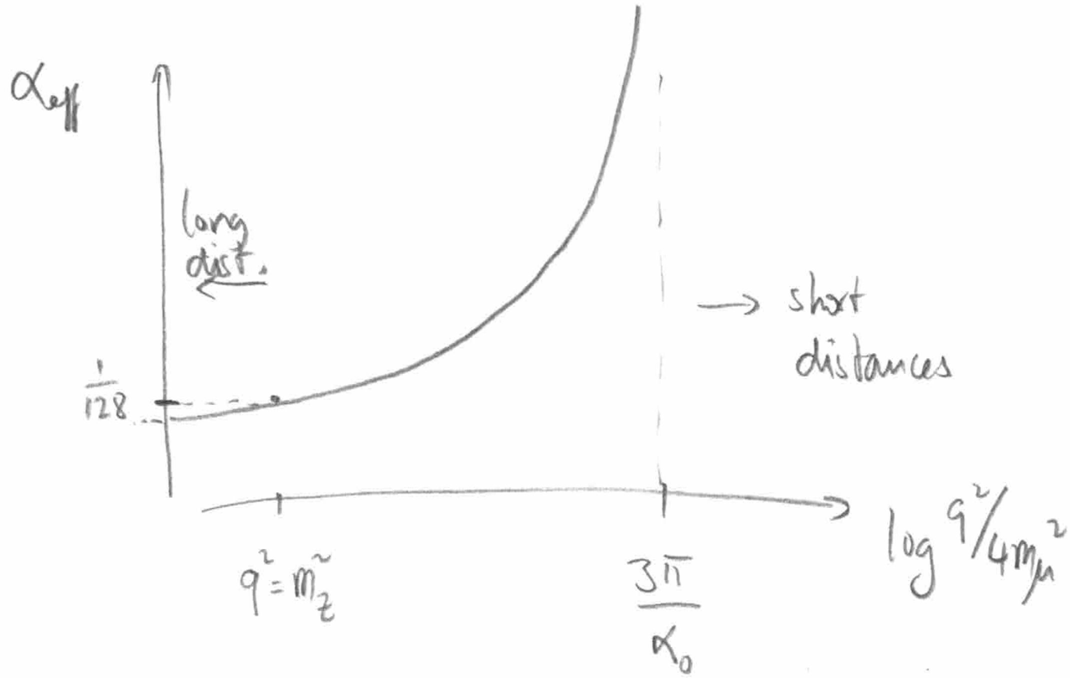
thus
$$M_{eff} = \frac{\text{spinors}}{p^2} \frac{e^2}{1 - \frac{\alpha}{3\pi}\log\frac{p^2}{4m_\mu^2}} \equiv \frac{\text{spinors}}{p^2} e_{eff}^2(p)$$

with $\boxed{\alpha_{eff} \equiv \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\log\frac{p^2}{4m_\mu^2}}}$

similarly, for t-channel



$$= \frac{u+s}{t} \frac{2e^2}{1 - \frac{\alpha}{3\pi}\log\frac{|q|^2}{4m_\mu^2}} = \frac{u+s}{t} 2e_{eff}^2$$



perturbation theory fails when expansion parameter $\gtrsim 1$

i.e. when $\frac{\alpha}{3\pi}\log\frac{|q|^2}{4m_\mu^2} \gtrsim 1$, this is much earlier

than naively expected (i.e. $\frac{\alpha}{4\pi} \simeq 1$) for $q^2 \gg m_\mu^2$.

We can avoid this premature fail by taking a derivative

RGE: $$\frac{d}{d\log q}\alpha(q) = q\frac{d}{dq}\alpha = \frac{\alpha_0}{\left(1-\frac{\alpha_0}{3\pi}\log\frac{|q|^2}{4m^2}\right)^2}\,2\frac{\alpha_0}{3\pi} = \frac{2}{3\pi}\alpha^2(q)$$

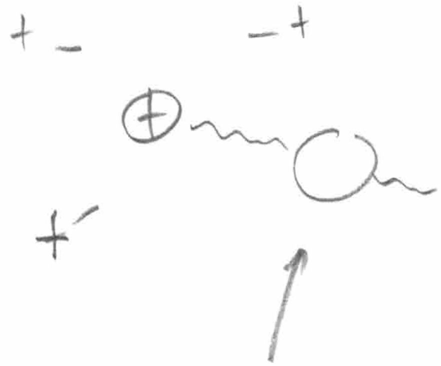
where $q \equiv \sqrt{|q|^2}$

or - using - $\bullet \equiv d/d\log q$

$$\boxed{\dot{\alpha} = \frac{2}{3\pi}\alpha^2}$$

diff's equation only breaks down when $\frac{2\alpha}{3\pi} \simeq 1$, i.e. no large logs.

Physics of RGE? screening due to polarization of vacuum



virtual charged dipoles screen the physical charge at larger distances

"vacuum polarization" Feynman diagram

multiple charged particles: $$\dot{\alpha}(q) = \sum_{\substack{f \\ 2m_f < q}} Q_f^2 \frac{2}{3\pi} \alpha^2(q)$$

solution? $$\dot{\left(\frac{1}{\alpha}\right)} = -\frac{\dot{\alpha}}{\alpha^2} = -\sum_f \frac{2Q_f^2}{3\pi} = \text{constant!}$$

$$\Rightarrow \quad \frac{1}{\alpha(q)} = \frac{1}{\alpha(q_0)} - \sum_f 2\frac{Q_f^2}{3\pi} \log q/q_0$$

when not crossing mass thresholds

$$\Rightarrow \quad \alpha(q) = \frac{\alpha(q_0)}{1 - \alpha(q_0)\frac{2}{3\pi}\sum Q_f^2 \log(q/q_0)}$$

Recap:

RGE QED: $$\dot{\alpha}(q) = \sum_{\substack{f \\ 2m_f < q}} \frac{2}{3\pi} Q_f^2 \alpha^2(q)$$

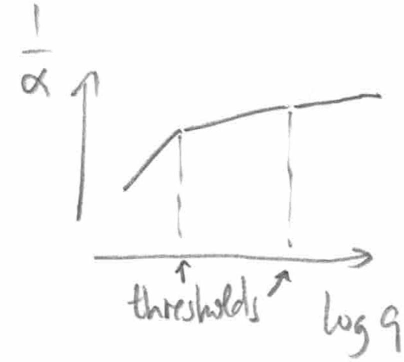
$$= \sum_{\substack{f_L, f_R \\ 2m_f < q}} \frac{1}{3\pi} Q_f^2 \alpha^2(q)$$

Weyl 2-component fermions.

solution: $$\dot{\left(\frac{1}{\alpha}\right)} = -\sum_f \frac{1}{3\pi} Q_f^2 = \text{const. (except at thresholds)}$$

$$\frac{d}{dt} = \frac{d}{d\ln q} \quad \Rightarrow \quad \frac{1}{\alpha} = \frac{1}{\alpha(q_0)} - \sum \frac{2}{3\pi} Q_f^2 \log(q/q_0)$$

$\Rightarrow$ piecewise linear in $\log q$
slope changes at thresholds



What's the advantage? this relates the coupling at q to the coupling at some reference q_0 which can be chosen near the experimental setup $\Rightarrow$ can avoid large logs in the theory. Ex LHC $q_0 \approx 100$ GeV $\Rightarrow$ only logs are $\log \frac{q_{LHC}}{100\,\text{GeV}}$ and not $\log \frac{q_{LHC}}{m_e}$.

real QED: $q < m_e \Rightarrow \dot{\alpha} = 0 \Rightarrow \alpha = \alpha_{em} = \frac{1}{137}$

$2m_e < q < 2m_\mu \qquad \frac{1}{\alpha} = 137 - \frac{2}{3\pi} \log q/2m_e$

$q = 2m_\mu \qquad \frac{1}{\alpha} = 137 - \frac{2}{3\pi} \log m_\mu/m_e = 135.9$

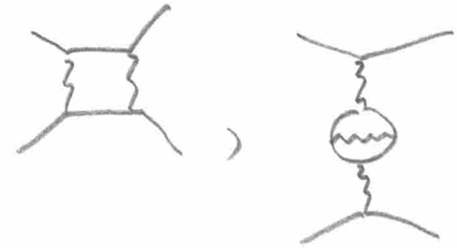
$q > m_\mu \qquad \frac{1}{\alpha} = 137 - \frac{2}{3\pi} \log q/m_e - \frac{2}{3\pi} \log q/m_\mu - \ldots$

$$q \simeq m_Z \qquad \frac{1}{\alpha(m_Z)} = 137 - \underbrace{\frac{2}{3\pi} \sum_f Q_f^2 \log \frac{m_Z}{m_f}}_{\approx 8} \approx 129$$

$\leftarrow$ includes 3x for color of quarks

pions from 100 MeV–GeV then quarks

$\alpha^{-1}(m_Z) = 127.9 \pm 0.1$ requires higher order corrections

e.g. 

What about above m_Z? EW symmetry restored

$$\Rightarrow \quad g_Y = \frac{e}{c_W} \qquad g_2 = \frac{e}{s_W} \qquad s_W = \sin\theta_W$$

Hyperchange $\qquad$ SU(2) $\qquad s_W^2 \approx 0.23$

$$\alpha_Y^{-1}(M_Z) = \alpha_{em}^{-1}(M_Z)\, c_W^2 = 128.77 = 99$$

so at what scale Λ does α_Y blow up? (this is called the "Landau pole")

$$\alpha_Y \to \infty \iff \frac{1}{\alpha_Y} = 0$$

$$\Rightarrow \frac{1}{\alpha_Y} = 0 = 99 - \frac{2}{3\pi} \log\frac{\Lambda}{M_Z} \cdot$$

$$\cdot\, 3\, \frac{1}{2}\Big[\underset{e_R}{1} + \underset{L_L}{2\left(\tfrac{-1}{2}\right)^2} + \underset{Q_L}{\overset{\text{color}}{3}\cdot 2\left(\tfrac{1}{6}\right)^2} + \underset{u_R}{3\left(\tfrac{2}{3}\right)^2} + \underset{d_R}{3\left(-\tfrac{1}{3}\right)^2}\Big] + \underset{H}{\tfrac{1}{4}\, 2\left(\tfrac{1}{2}\right)^2}$$

generations $\quad$ Weyl fermions $\qquad\qquad$ complex scalar

$$\Rightarrow \log \Lambda/M_Z = \frac{3\pi}{2}\, 99\, \frac{8}{41} = 91 \Rightarrow \Lambda \sim 10^{41}\,\mathrm{GeV} \gg M_{pe} = 10^{19}\,\mathrm{GeV}$$

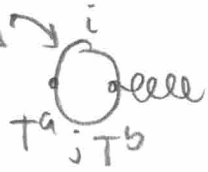
QCD?

QED

charged fermions

$$\rightarrow Q^2 \frac{2\alpha}{3\pi} \log q/2m \rightarrow \dot\alpha = \sum_f Q_f^2 \frac{2}{3\pi}\alpha^2$$

quarks

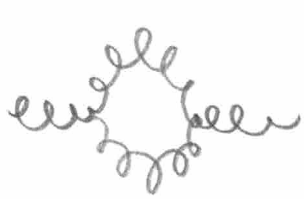


$$\rightarrow \frac{1}{2} \frac{2\alpha}{3\pi} \log q/2m \rightarrow \dot\alpha_s = \sum_f \frac{1}{3\pi}\alpha_s^2$$

$\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$

new ingredient... gluons carry color

$\Rightarrow$



Nobel for asymptotic freedom

$$\dot\alpha_s = \frac{\alpha_s^2}{3\pi} N_q - \frac{\alpha_s^2}{2\pi}\frac{11}{3} N_c < 0$$

$N_c = 3$

$N_q = 6$

Color field anti-screens

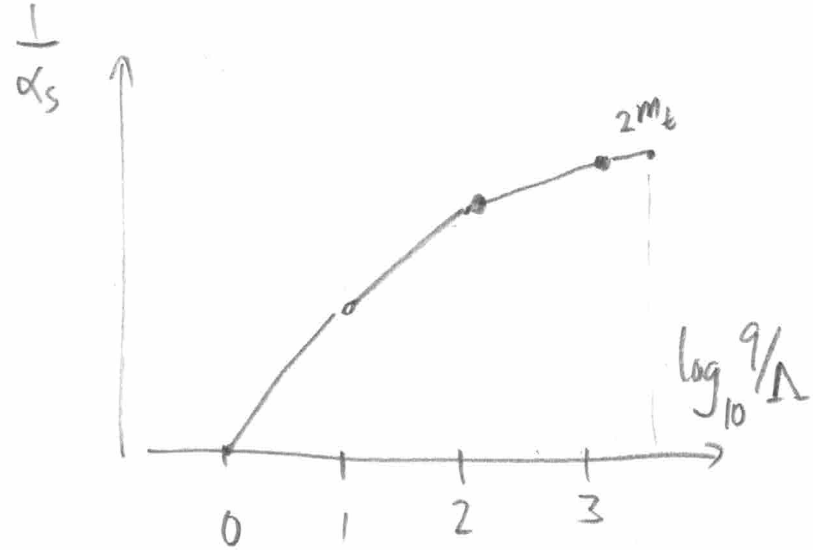
$\sim f_{abc} = \epsilon_{abc}$

end up getting $-$ sign

$f_{abc} f_{bcd}$

$$\frac{1}{\alpha_s(q)} = \frac{1}{\alpha_s(q_0)} + \frac{11 - \frac{2}{3}N_q}{2\pi} \log q/q_0$$

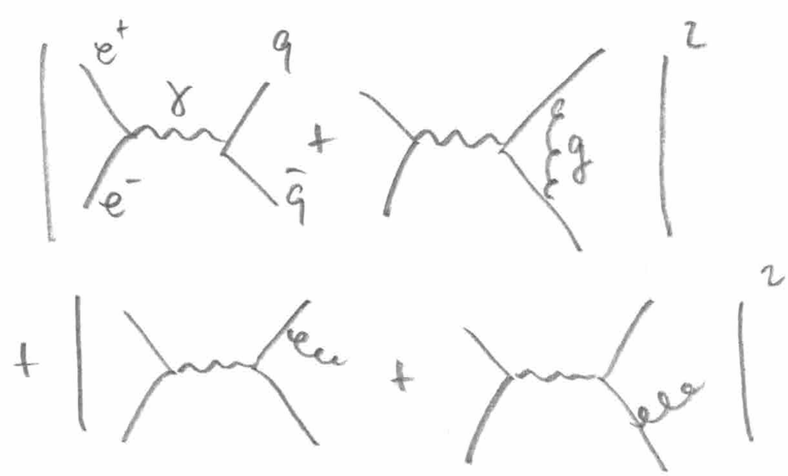
q	α_s	N_q
m_Z	0.118	
		$\downarrow$ 5
10 GeV	0.18	
		$\downarrow$ 2-4
1 GeV	0.35	
Λ 0.1 GeV	$-\infty$	0



$\alpha_s(\infty) \to 0$ asymptotic freedom

Measuring α_s?

1. $e^+e^- \to$ hadrons



$$= \frac{4\pi}{3s}\alpha_{em}^2 \sum_f Q_f^2 N_c \left[1 + \frac{\alpha_s\left(\frac{E_{CM}}{2}\right)}{\pi} + \ldots\right]$$

e.g. Z-pole $E_{CM} = M_Z$

2. $3j/2j$ ratio

3. spectroscopy of $b\bar{b}$ bound states

Running of all SM gauge couplings $SU(3)_c\, SU(2)_W\, U(1)_Y$

$$\dot{\alpha}_3 = -\frac{\alpha_3^2}{2\pi}\left(\underset{\uparrow_1}{11\frac{N_c}{3}} - \frac{2}{3}\underset{\uparrow\,6}{N_f}\right) = -\frac{\alpha_3^2}{2\pi}\,7$$

$$\dot{\alpha}_2 = -\frac{\alpha_2^2}{2\pi}\left(\frac{22}{3} - \frac{1}{3}\underset{\uparrow}{N_2} - \frac{1}{6}\overset{\text{1 Higgs doublet}}{N_H}\right) = -\frac{\alpha_2^2}{2\pi}\frac{19}{6}$$

SU(2) doublet fermions $= 3\cdot(\underset{L}{1}+\underset{Q}{3}) = 12$

$$\dot{\alpha}_1^2 = -\frac{\alpha_1^2}{2\pi}\left(0 - \frac{2}{3}\sum_f Q_f^2 - \frac{1}{3}\sum_h Q_h^2\right) = +\frac{\alpha_1^2}{2\pi}\frac{41}{6}$$

$$\dot{\alpha}_i = -\frac{\alpha_i^2}{2\pi} b_i \qquad b_i^{SM} = \left(-\frac{41}{6}, \frac{19}{6}, 7\right)$$

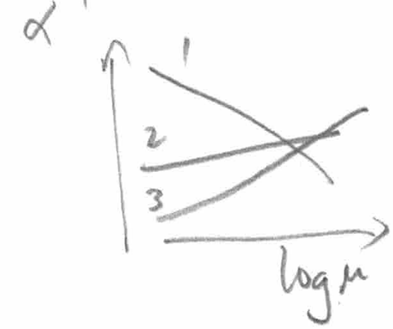
solution? $\left(\dfrac{1}{\alpha_i}\right)^{\!\cdot} = \dfrac{b_i}{2\pi}$

$$\Rightarrow \frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(M_Z)} + \frac{b_i}{2\pi}\log\left(\mu/M_Z\right)$$

$$\alpha_i^{-1}(M_Z) = \left(98.369 \pm 0.004,\ 29.586 \pm 0.004,\ 8.47 \pm 0.01\right)$$

PDG 2019

PLOTs from Mathematica SM_RGE_2020.nb



intriguing that couplings approximately unify but we've made a error, the hypercharge normalization is arbitrary $\Rightarrow$ α_1^{-1} line has arbitrary shift in the vertical direction.

To see this let's look at the covariant derivative ...