$$D_\mu = \partial_\mu + i g_2 W^a_{\mu_{SU(2)}} T^a_{SU(2)} + i g_3 G^a_{\mu_{SU(3)}} T^a_{SU(3)} + i g_1 B_\mu Q_Y$$

normalization?

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

normalization? e_R^- charge$^{U(1)_Y}$ $= -1$

what is the equivalent normalization if $U(1)_Y$ comes from non-Abelian group

<u>Example 1.</u> $U(1)_{\text{isospin}}$ inside $SU(2)_W$

$$T^a = \frac{\sigma^a}{2}, \quad \frac{\sigma^3}{2} \text{ is the diagonal generator}$$

$$i g_2 W^3_\mu \frac{\sigma^3}{2} \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \text{charges are } \pm \tfrac{1}{2}$$

Example 2: $U(1)$'s inside $SU(3)$

2 diagonal generators: $T_3 = \begin{pmatrix} \sigma_3/2 & \\ & 0 \end{pmatrix}$

$$T_8 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

acting on triplets: $i g_3 G^3_\mu T^3 = i\, g_3 G_\mu \begin{pmatrix} 1/2 & & \\ & -1/2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

charges $\pm 1/2$

$$i g_3 G^8_\mu T^8 = i\, g_3 G_\mu \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \begin{pmatrix} q \end{pmatrix}$$

charges $1/\sqrt{12}$, $-1/\sqrt{3}$

lesson $U(1)$ normalization not unique, depends on how $U(1)$ is embedded in a non-Abelian group.

$\Rightarrow$ need to specify unification group before proceeding.

Example 3: $SU(5)$:

want unification group that contains $SU(3)_c \times SU(2)_w \times U(1)_Y$

$SU(5)$ (special) unitary 5×5 matrices $\quad U = e^{i\alpha_a T^a}$

T^a: 5×5 herm.

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

basis: T^a, $a = 1 \ldots 24$

$$SU(3)_c: \begin{pmatrix} \begin{matrix} 3\times3 \\ 8\,T^a \end{matrix} & \begin{matrix} 3\times2 \\ B \end{matrix} \\ B^\dagger & \begin{matrix} 2\times2 \\ 3\,T^a \end{matrix} \end{pmatrix} \leftarrow SU(2)_w \qquad B = \begin{pmatrix} 1, i & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{2} \qquad T^a_B = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}$$

12 of these

$\Rightarrow$ have 23 generators, T^a_B do not commute with $SU(3)$, $SU(2)$

$\Rightarrow$ cannot be $U(1)_Y$

missing one generator, look at diagonal ones.

there should be a total of $5 - \underset{\uparrow\ \text{traceless}}{1} = 4$.

Have T_3, T_8 inside $SU(3)_c$, T_3 inside $SU(2)_W$

$$\left(\begin{array}{ccc|cc} 1 &&&& \\ & -1 &&& \\ && 0 && \\ \hline &&&& \\ &&&& \end{array}\right) \left(\begin{array}{ccc|cc} 1 &&&& \\ & 1 &&& \\ && -2 && \\ \hline &&&& \\ &&&& \end{array}\right) \left(\begin{array}{ccc|cc} &&&& \\ &&&& \\ &&&& \\ \hline &&& 1 & \\ &&&& -1 \end{array}\right) \quad \text{missing 1!}$$

something that commutes with $SU(3)$ & $SU(2)$ must be

$$\left(\begin{array}{ccc|cc} a &&&& \\ & a &&& \\ && a && \\ \hline &&& b & \\ &&&& b \end{array}\right), \text{traceless} \quad N \begin{pmatrix} -\frac{1}{3} &&&& \\ & -\frac{1}{3} &&& \\ && -\frac{1}{3} && \\ &&& \frac{1}{2} & \\ &&&& \frac{1}{2} \end{pmatrix} = T_{24}$$

normalization $\mathrm{Tr}(T_{24} T_{24}) = \frac{1}{2} = \left(3 \cdot \frac{1}{9} + 2 \frac{1}{4}\right) N^2 = N^2 \frac{5}{6}$

$$\Rightarrow N = \sqrt{3/5}$$

How do fermions couple? 5-plets of fermions

$$5 = \begin{pmatrix} \\ \\ \hdashline \\ \\ \end{pmatrix} \begin{array}{l} \leftarrow 3 \text{ of color } SU(3) \\ \\ 2 \text{ of weak } SU(2) \end{array}$$

$$i g_5 T^a A^a_\mu \begin{pmatrix} \\ \cdots \\ \\ \end{pmatrix} \Rightarrow \text{upper components triplet of } SU(3)_c$$

on triplet $\Rightarrow \Big(i g_5 \underset{\substack{\uparrow \\ 1\ldots 8}}{T_3^a} G^a_\mu + i \underbrace{g_5\sqrt{3/5}}_{g_Y} \underbrace{(-1/3)}_{Q_Y} B_\mu \Big) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \qquad (3,1)_{-1/3}$

on doublet $\Big(i g_5 T_2^a W^a_\mu + i \underbrace{g_5\sqrt{3/5}}_{g_Y} \underbrace{1/2}_{Q_Y} B_\mu \Big) \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \qquad (1,2)_{1/2}$

note that $\bar{5}$ transforms like $(\bar{3},1)_{1/3} + (1,2)_{-1/2}$

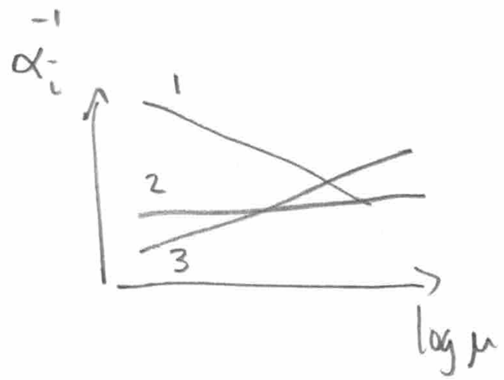
$\uparrow$ complex conjugate of 5

Recap

RGEs : $\dot{\alpha}_i = -\frac{\alpha_i^2}{2\pi} b_i \qquad b_i^{SM} = \left(-\frac{41}{6}, \frac{19}{6}, 7\right)$

$U(1) \quad SU(2) \quad SU(3)$

solution



normalization of coupling of $U(1)$?

$$\partial_\mu + i g A_\mu^a T^a + i g_1 B_\mu Q_Y$$

$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

$Q_{EM}(e^-) = -1$

need to embed $U(1)$ into non-abelian group to understand normalization.

$SU(5)$

$$T^a = \left(\begin{array}{c|c} 8 \times T^a_{SU(3)} & \begin{matrix} 1, i \\ 12 \end{matrix} \\ \hline \begin{matrix} 1, -i \\ 12 \end{matrix} & 3 \times T^a_{SU(2)} \end{array}\right) \qquad \mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$$

$$T^{24} = \sqrt{3/5} \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix}$$

commutes with $SU(3)$ & $SU(2)$

How do fermions couple?

guess that they are in a 5-dim representation i.e. $5 = \begin{pmatrix} \\ \\ \end{pmatrix}$ } 5 comp. transform under $SU(5)$

$\begin{pmatrix} \\ \hline \\ \end{pmatrix}$ ← $SU(3)$ ⇒ color triplet !
← $SU(2)$ ⇒ $SU(2)_W$ doublet !

$$i g_5 T^a A_\mu^a \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \hline \cdot \\ \cdot \end{pmatrix}$$

on triplet get: $i g_5 T^a_{SU(3)} G^a_\mu + i g_5 \sqrt{3/5}\left(-\frac{1}{3}\right) B_\mu \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad (3,1)_{-1/3}$

on doublet $\quad i g_5 T^a_{SU(2)} W^a_\mu + \underbrace{i g_5 \sqrt{3/5}}_{\equiv g_Y} \underbrace{\tfrac{1}{2}}_{Q_Y} B_\mu \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \quad (1,2)_{1/2}$

↑ absorb $\sqrt{3/5}$ into def. of coupling to get "normal looking" hypercharges

notation "direct sum" $5 = (3,1)_{-1/3} \oplus (1,2)_{1/2}$

note: $\bar{5} = (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}$

$\bar{2} = 2$

SM fermions $Q = (3,2)_{1/6}$

$U^c = (\bar{3}, 1)_{-2/3}$

$D^c = (\bar{3}, 1)_{1/3}$

$L = (1, 2)_{-1/2}$

$E^c = (1, 1)_1$

$\Rightarrow$ D^c, L have just the right QNs to unify into $\bar{5}$

normalizations of couplings: unification predicts

$g_2 = g_3 = g_5$

$g_Y = \sqrt{3/5}\, g_5$

$\Rightarrow$ should expect unification of $g_2, g_3, \sqrt{5/3}\, g_Y$

$\Leftrightarrow$ $\alpha_2^{-1}, \alpha_3^{-1}, \alpha_1^{-1} \frac{3}{5}$

$\longrightarrow$ mathematica notebook

"unification" scale   SM $10^{14}, 10^{15}, 10^{17}$ GeV

SUSY $2 \cdot 10^{16}$ GeV

What about the other fermions?

$$5 = (3\ 1)_{-1/3} + (1\ 2)_{1/2}$$

$$\bar{5} = (\bar{3}\ 1)_{1/3} + (1\ 2)_{-1/2}$$

other representations   example SU(2): $\frac{1}{2} \times \frac{1}{2} = 0 + 1$

$$\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = (0+1) \times \tfrac{1}{2}$$

$$= \tfrac{1}{2} + \left(\tfrac{1}{2} + \tfrac{3}{2}\right)$$

in our notation $2 \times 2 = 1 + 3$

$$2 \times 2 \times 2 = (1+3) \times 2$$

$$= 2 + (2 + 4)$$

SU(3): $3 \times \bar{3} = \underset{\text{singlet}}{1} + \underset{\text{gluons}}{8}$

$$3 \times 3 = \underset{\text{antisymm}}{\bar{3}} + \underset{\text{symmetric}}{6}$$

$$SU(5): \quad 5 \times \bar{5} = 1 + 24$$

singlet   gauge bosons

recall 24 generators of $SU(5)$

$$5 \times 5 = \underset{AS}{10} + \underset{S}{15}$$

$$10 = 5\times 5\Big|_A = \Big[\big((3,1)_{-1/3} + (1,2)_{1/2}\big) \times \big((3,1)_{-1/3} + (1,2)_{1/2}\big)\Big]_A$$

$$= (\bar{3},1)_{-2/3} + (1,1)_1 + (3,2)_{+1/6}$$

$$= U^c + E^c + Q \quad !!!$$

$$\text{One generation of SM} = \underbrace{D^c \; L}_{\bar{5}} \; \underbrace{U^c \; E^c \; Q}_{10}$$

predictions? unification of couplings

- gauge couplings
- Yukawa couplings

$$Q U^c H \qquad Q D^c \tilde{H} \qquad E^c L \tilde{H}$$

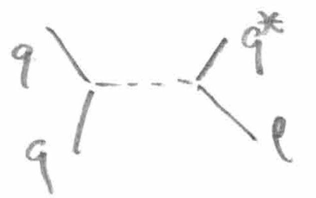
$$10\ 10\ 5 \qquad 10\ \bar{5}\ \bar{5} \qquad 10\ \bar{5}\ \bar{5}$$

$\Rightarrow$ H is in a 5 rep $\Rightarrow$ $\begin{matrix}(\bar{3},1)_{-1/3} \\ \end{matrix} \begin{pmatrix} H_3 \\ \hline H_2 \end{pmatrix}$ $\leftarrow$ colored partner of Higgs.

$(1,2)_{1/2}$

H_3 mass must be at GUT-scale in order for our RGE calculations to be valid (we assumed only a Higgs doublet)

Also, Higgs triplet mediates proton decay



$\sim \dfrac{1}{M_{H_3}^2}$ $\Rightarrow$ need H_3 at GUT scale.

Recap $\quad SU(5)$ contains $SU(3)\times SU(2)\times U(1)_Y$ as subgroup

generators $SU(5)$: Hermitian traceless 5×5

$$A_\mu = A_\mu^a T^a \sim \left(\begin{array}{c|c} G_\mu & XY_\mu^\dagger \\ \hline XY_\mu & W_\mu \end{array}\right) + \sqrt{\tfrac{3}{5}}\, B_\mu \left(\begin{array}{c|c} -1/3 & \\ \hline & +1/2 \end{array}\right)$$

i.e. $24 \longrightarrow (8,1)_0 + (1,3)_0 + (1,1)_0 + (3,2)_? + (\bar{3},2)_{-?}$

what is hypercharge of XY?

recall $\quad 5 \longrightarrow (3,1)_{-1/3} + (1,2)_{1/2}$

$\qquad\quad \bar{5} \longrightarrow (\bar{3},1)_{1/3} + (1,2)_{-1/2}$

$$\Rightarrow 5\times\bar{5} = \left((3\ 1)_{-1/3} + (1\ 2)_{1/2}\right)\times\left((\bar{3}\ 1)_{1/3} + (1\ 2)_{-1/2}\right)$$

$$= \underbrace{(1\ 1)_0}_{1} + \underbrace{(8\ 1)_0 + (1,1)_0 + (1,3)_0 + (3,2)_{-5/6} + (\bar{3},2)_{5/6}}_{24}$$

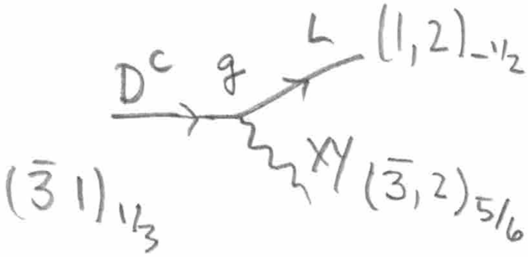
Vector boson: $(\bar{3},2)_{+5/6} \xrightarrow[\text{breaks}]{SU(2)\times U(1)_Y} 3_{+4/3} + 3_{+1/3}$

What does it couple to?

$$\bar{5}^{\dagger} i\bar{\sigma}^{\mu} D_{\mu}\, 5 \to \bar{5}^{\dagger}\, \bar{\sigma}^{\mu} g A_{\mu}\, \bar{5}$$

$$\bar{5} = \begin{pmatrix} D^c \\ L \end{pmatrix}$$

$$\Rightarrow g\left[D^{c\dagger}\bar{\sigma}\cdot G D^c + L^{\dagger}\bar{\sigma}\cdot W L + \left(D^{c\dagger}\bar{\sigma}\cdot XY\, L + h.c.\right) + \text{hypercharge} \right]$$

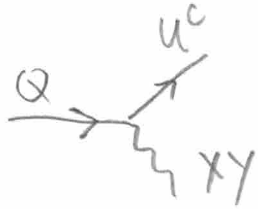


$$\left((\bar{3}\;1)_{+1/3}\right)^{\dagger} (\bar{3}.2)_{+5/6}\, (1\,2)_{-1/2}$$

couplings to 10?

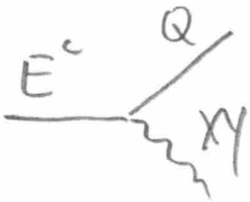
$$10 = 5\times 5_A = (\bar{3},1)_{-2/3} + (1,1)_1 + (3,2)_{1/6}$$

$$10^{\dagger}\bar{\sigma}^{\mu} A_{\mu}\, 10 \longrightarrow \underset{Q^{\dagger}}{(\bar{3},2)_{-1/6}}\; \underset{XY}{(\bar{3},2)_{5/6}}\; \underset{u^c}{(\bar{3},1)_{-2/3}}$$

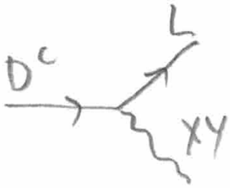


$(1\ 1)_{-1}$ $(\bar{3}\ 2)_{5/6}$ $(3\ 2)_{1/6}$

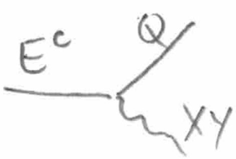
$E^{c\dagger}$ XY Q



What is Baryon and Lepton # of XY?



$\Rightarrow$ $B = -1/3$ $L = -1$ $B-L = 2/3$



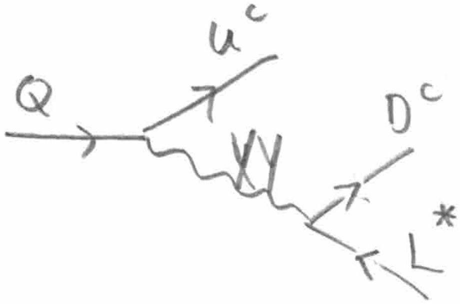
$\Rightarrow$ $B = -1/3$ $L = -1$ $B-L = 2/3$

$\Rightarrow$ $B = 2/3$ $L = 0$ $B-L = 2/3$

$\Rightarrow$ no consistent assignment of B & L possible

$\Rightarrow$ Baryon # and L # violation!

But B-L conserved. ⇒ processes which preserve B-L allowed



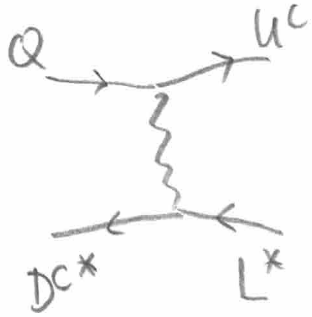
$$M \sim \frac{g^2}{M_{XY}^2} Q^\dagger \bar{\sigma}^\mu u^c D^c \bar{\sigma}_\mu L^\dagger$$

$q \to \bar{q}\bar{q}\bar{\ell}$ B-L ok

$\Delta B = -1$ $\Delta L = -1$

$\cancel{uud \to \bar{u}\bar{d}e^+ ud}$

or $qq \to \bar{q}\bar{\ell}$



$$uud \xrightarrow{X} u u^c e^+ \qquad \pi^0 e^+$$
$$\xrightarrow{Y} u d^c \bar{\nu} \qquad \pi^+ \bar{\nu}_e$$

Rate?
$$M \sim \frac{g_5^2}{M_{XY}^2} \Rightarrow \Gamma \sim \frac{m_p^5}{8\pi} \frac{g_5^4}{M_{XY}^4}$$

g_5? unified coupling

M_{XY}? unification scale

$$g_5 \sim 1 \,, \qquad \Gamma \sim \frac{\mathrm{GeV}^5}{8\pi} \frac{1}{(2\,10^{16}\,\mathrm{GeV})^4} \sim 10^{-67}\,\mathrm{GeV}$$

$$\frac{\mathrm{GeV}\;10^{-15}\,\mathrm{m}}{10^{9}\,\mathrm{m/s}} = \mathrm{GeV}\;10^{-24}\,\mathrm{s}$$

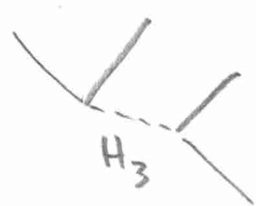
$$\tau \sim 10^{67}\;10^{-24}\,\mathrm{s} \sim 10^{43}\,\mathrm{s} \sim 10^{36}\,\mathrm{yrs}$$

Yukawa couplings

$$10\;10\;5 \nwarrow H \rightarrow \begin{pmatrix} (3\;1)_{-1/3} \\ (1\;2)_{1/2} \end{pmatrix} \begin{array}{l} H_3 \\ \leftarrow H_2 \end{array}$$

$$10\;\bar{5}\;\bar{5} \nwarrow H^c$$

H_3 also mediates proton decay



$$M \sim \frac{\lambda^2}{M_{H_3}^2} \Rightarrow M_{H_3} \text{ also large.}$$