CH 6: Lagrangians

derive $j_\mu j^\mu$ interaction from generalization of Maxwell eqs for spin 1 particles.

Want: -

- local interactions $\leftrightarrow$ simple diff.eq.

  (causality requires locality)

  i.e. not $\theta_\mu \partial^\mu \phi(x) = \int dy\ f(x-y)\ j(y)$

  non-local

- lorentz invariance

  i.e. $A_\mu(x)$ solution to EOM $\Rightarrow$ $A'_\mu = \Lambda_\mu{}^\nu A_\nu$ also soln.

  $\Rightarrow$ physics is frame independent

use Lagrangian formalism.

Action: $S = \int dt\ L = \int d^4x\ \mathcal{L}(\vec{x}, t)$

$$\mathcal{L}\left[\text{fields}(x),\ \partial_\mu(\text{fields})\right]$$

what fields?

spin 0 : scalar $\phi(x)$

spin 1 : vector $A^\mu(x)$

spin ½ : 2-component spinor $\psi_\alpha(x)$ $\alpha = 1, 2$

Dirac fermion? $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ ← 2 2-component spinors

Example: real scalar $\phi(x)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right) \qquad \phi(\overset{\text{implied}}{x})$$

variational principle: $\phi \to \phi + \delta\phi$

postulate S stationary $\Longleftrightarrow$ $\delta S = 0$

$$\delta S = \int d^4x \;\; \partial_\mu(\delta\phi)\,\partial^\mu\phi - m^2 \delta\phi\,\phi$$

$$= \int d^4x \; \delta\phi\left(-\partial_\mu\partial^\mu\phi - m^2\phi\right) + \text{B.T.}$$

assume B.C.s s/th BT = 0

$$\overset{!}{=} 0 \quad \text{for all } \delta\phi$$

$$\Rightarrow \quad (\partial^2 + m^2)\,\phi = 0 \qquad \text{Klein-Gordon-EOM}$$

complex scalar ϕ

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \qquad \text{hermitian}$$

$$0 = \delta S = \int d^4x \; \delta\phi^* \left(-\partial^2\phi - m^2\phi\right) + \delta\phi \left(-\partial^2 - m^2\right)\phi^*$$

$$\Rightarrow \quad (\partial^2 + m^2)\,\phi = 0 \qquad + \text{h.c.}$$

symmetries: $\mathcal{L}$ invariant under "global" symmetry transformation

$$\phi(x) \to e^{i\alpha}\phi(x)$$
$$\phi^* \to e^{-i\alpha}\phi^*$$

Noether's theorem: global symmetry $\Longrightarrow$ conserved current $j^\mu(x)$

$$j^\mu(x) = (\rho(x), \vec{j}(x))$$

conserved: $\partial_\mu j^\mu = 0 = \dot{\rho} + \vec{\nabla}\cdot\vec{j}$

note $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla})$

$\partial^\mu = (\partial_t - \vec{\nabla})$ opposite to ordinary vectors

$x^\mu = (t, \vec{x})$
$x_\mu = (t, -\vec{x})$

$$(\partial_\mu x^\nu = \delta_\mu^\nu)$$

$$Q \equiv \int d^3x\, \rho(x)\,, \quad \dot{Q} = \int d^3x\, \dot{\rho} = -\int d^3x\, \vec{\nabla}\cdot\vec{j}$$

$$= BT = 0$$ as long as sources vanish at ∞.

$\boxed{\text{Insert } 81'}$

here: $j^\mu = -i\left[\phi^* \partial^\mu \phi - \partial^\mu(\phi^*)\,\phi\right]$

check conservation: $i\partial_\mu j^\mu = \phi^*\partial^2\phi + \cancel{\partial_\mu\phi^*\partial^\mu\phi} - \cancel{\partial^\mu\phi^*\partial_\mu\phi} - \partial^2(\phi^*)\phi$

EOM

$$= -m^2\phi^*\phi - (-m^2\phi^*\phi) = 0$$

i.e. $\partial_\mu j^\mu = 0$ on solutions to EOM.

2-component fermion: Dirac 4-comp. $\begin{pmatrix} \psi_L \\ \hline \psi_R \end{pmatrix}$ ← this one

$$\mathcal{L} = \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \sigma^\mu = (\mathbb{1}\ \vec{\sigma}) \quad \bar{\sigma}^\mu = (\mathbb{1}\ -\vec{\sigma})$$

EOM: $i\bar{\sigma}^\mu \partial_\mu \psi = 0$

guess plane wave $\quad \psi = \xi\, e^{-ip_\mu x^\mu} \qquad \partial_\mu = \dfrac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla})$

Noether also tells us what the current is:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi)$$

Symmetry: $\phi \to e^{i\alpha}\phi \cong \phi + i\alpha\phi \equiv \phi + \delta\phi$

$(\delta\mathcal{L} = 0)$ $\quad \phi^* \to e^{-i\alpha}\phi^* \equiv \phi^* + \delta\phi^*$

more general: local transformation $\phi \to \phi + \delta_{local}\phi \equiv \phi + i\alpha(x)\phi$

$$\delta_{local}\phi^* = \phi - i\alpha(x)\phi^*$$

$$\delta\mathcal{L}_{local} = \overset{=0}{\cancel{\delta\mathcal{L}_{global}}} + \underbrace{\text{terms with } \partial_\mu\alpha}$$

$$= i\left(\partial_\mu\phi^* \partial^\mu\alpha\, \phi - \partial_\mu\alpha\, \phi^* \partial^\mu\phi\right)$$

$$= i\, \partial_\mu\alpha \underbrace{\left((\partial^\mu\phi^*)\phi - \phi^*\partial^\mu\phi\right)}_{\equiv j^\mu}$$

but if ϕ = soln to EOM, then $\delta S = \int d^4x\, \delta\mathcal{L}_{local} = 0$

$$\Rightarrow \quad 0 = \int d^4x\, \alpha(x)\, \partial_\mu j^\mu$$

α arbitrary $\Rightarrow \partial_\mu j^\mu(x) = 0$ $\quad$ conserved current.

$$\bar{\sigma}^\mu p_\mu \xi = \bar{\sigma}\cdot p\, \xi = 0$$

$$\Rightarrow \sigma\cdot p\, \bar{\sigma}\cdot p\, \xi = (p^0 - \vec{p}\cdot\vec{\sigma})(p^0 + \vec{p}\cdot\vec{\sigma})\xi = (p^{0\,2} - \vec{p}^{\,2})\xi = 0$$

$\Rightarrow E = |\vec{p}|$ massless dispersion

choose $\vec{p} \parallel \hat{z}$ $\Rightarrow E(1+\sigma_3)\xi = 2E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\xi = 0 \Rightarrow \xi = \begin{pmatrix}0\\1\end{pmatrix}$

massless, <u>left-helicity</u> spin ½ $\left(\frac{\vec{\sigma}}{2}\cdot\hat{p}\,\xi = -\frac{1}{2}\xi\right)$

right-helicity: $\mathcal{L} = \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R$

$$\Rightarrow i\sigma^\mu\partial_\mu\psi_R = 0 \;\ldots \Rightarrow \xi = \begin{pmatrix}1\\0\end{pmatrix}$$

massive spin ½ : need both helicities

$$\mathcal{L} = \psi_R^\dagger i\sigma\cdot\partial\psi_R + \psi_L^\dagger i\bar{\sigma}\cdot\partial\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

4-component notation: $\Psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$, $\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$

EOM: $(i\partial\!\!\!/ - m)\Psi = 0$ Dirac Eq.

plane wave solutions: $(\not{p}-m)u_p=0$

$$(\not{p}+m)(\not{p}-m)u_p=(\not{p}^2-m^2)u_p=(p^2-m^2)u_p$$

$\Rightarrow$ $E^2=\vec{p}^2+m^2$ massive dispersion.

<u>spin 1</u>: complex ϕ $\quad \phi\to e^{i\alpha}\phi$ global symmetry

$$\mathcal{L}=\partial^\mu\phi^*\partial_\mu\phi-V(\phi^*\phi) \quad \text{invariant}$$

what about <u>local</u> $\alpha(x)$ phase rotations? $\quad V$ invariant

$$\partial_\mu\phi\to e^{i\alpha(x)}\left(\partial_\mu+i\,\partial_\mu(\alpha)\right)\phi$$

$$\Rightarrow \partial^\mu\phi^*\partial_\mu\phi\to(\partial^\mu-i\partial^\mu\alpha)\phi^*(\partial_\mu+i\partial_\mu\alpha)\phi \quad \text{not invariant}$$

trick: introduce vector field $A_\mu(x)\to A_\mu(x)+\frac{1}{g}\partial_\mu\alpha(x)$

$\Rightarrow$ $D_\mu\phi\equiv(\partial_\mu-igA_\mu)\phi$ transforms nicely

check: $\partial_\mu\phi-ieA_\mu\phi\to e^{i\alpha}(\partial_\mu+i\cancel{\partial_\mu\alpha})\phi-ig(A_\mu+\frac{1}{g}\cancel{\partial_\mu\alpha})e^{i\alpha}\phi$

$$=e^{i\alpha}D_\mu\phi$$