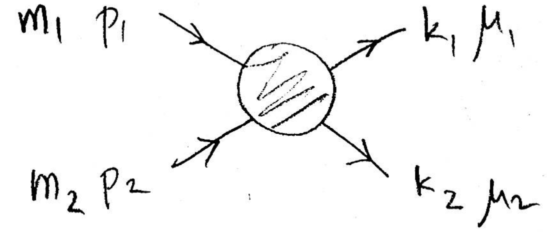
$e^+e^- \to \mu^+\mu^-$ scattering




$$k_2 = p_1 + p_2 - k_1$$

if external particles are helicity eigenstates, then the Lorentz invariant $\mathcal{M}$ can only depend on $p_i^2 = m_i^2$, $\underbrace{p_1 \cdot p_2,\ p_1 \cdot k_1,\ p_2 \cdot k_1}_{\text{3 L. Inv. fctns}}$

def: <u>Mandelstam variables</u>

$$s = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2$$

$$t = (k_1 - p_1)^2 = \mu_1^2 + m_1^2 - 2\,p_1 \cdot k_1$$

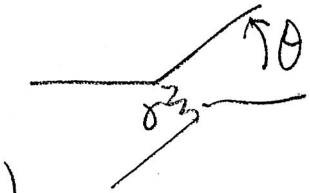
$$u = (k_1 - p_2)^2 = \mu_1^2 + m_2^2 - 2\,p_2 \cdot k_1$$

constraint: 
$$\mu_2^2 = (p_1+p_2)^2 - 2(p_1+p_2)k_1 + k_1^2$$
$$= s + t + u - m_1^2 - m_2^2 - 2\mu_1^2 + \mu_1^2$$
$$\Rightarrow \quad s + t + u = \sum_i^4 m_i^2$$

$$\mathcal{M}(\text{masses}, s, t, u)$$

in CM; all massless particles

$p_1 = E(1\,0\,0\,1)$ $\quad p_2 = E(1\,0\,0\,-1)$



$k_1 = E(1\ \sin\theta\ 0\ \cos\theta)$ $\quad k_2 = E(1\ -\sin\theta\ 0\ -\cos\theta)$

$$s = (2E)^2 = E_{CM}^2$$

$$t = -2p_1\cdot k_1 = \frac{-E_{CM}^2}{2}(1-\cos\theta)$$

$$u = -2p_2\cdot k_1 = -\frac{E_{CM}^2}{2}(1+\cos\theta)$$

$$\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = e^2(1+\cos\theta) = -2e^2\frac{u}{s} = 4e^2\frac{p_2\cdot k_1}{(p_1+p_2)^2}$$

$$\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+) = e^2(1-\cos\theta) = -2e^2\frac{t}{s} = 4e^2\frac{p_1\cdot k_1}{(p_1+p_2)^2}$$

understanding the denominator:

photon propagator $\gamma$ $\sim \frac{1}{(p_\gamma)^2} = \frac{1}{(p_1+p_2)^2}$

photon coupling to source $eA_\mu J^\mu$

wave equation of E&M potential: $\left[\partial_t^2 - \vec{\nabla}^2\right] A^\mu(x) = eJ^\mu(x)$

fourier space $-((k^0)^2 - \vec{k}^2)\tilde{A}^\mu(k) = e\tilde{J}^\mu(k)$

$$\tilde{A}^\mu(k) = \frac{-e}{k^2}\tilde{J}^\mu(k)$$

$\tilde{J}^\mu(k)$ is current corresponding to $e_R^- e_L^+$ source.

$\Rightarrow$ $k = p_1 + p_2$ momentum of $\gamma$.

massive wave equation: $\left(\partial_t^2 - \vec{\nabla}^2 + m^2\right) A^\mu = e J^\mu$

$$\Rightarrow \quad \tilde{A}^\mu = -\frac{e}{s-m^2} \tilde{J}^\mu$$

massive propagator
(note resonance when $s = m^2$)

__numerators__? (spin overlap)

initial $e_R^-$ (helicity) $\quad u_p^+ = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^+ \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^+ \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\ \xi^+ \\ \sqrt{E+p}\ \xi^+ \end{pmatrix} \overset{m=0}{=} \sqrt{2E}\begin{pmatrix} 0 \\ \xi^+ \end{pmatrix}$

+ helicity

using $p\cdot\sigma\,\xi^+ = (E - \vec{p}\cdot\vec{\sigma})\xi^+ = (E-p)\xi^+$

$$u_p^- = \begin{pmatrix} \sqrt{E+p}\ \xi^- \\ \sqrt{E-p}\ \xi^- \end{pmatrix} \overset{m=0}{=} \sqrt{2E}\begin{pmatrix} \xi^- \\ 0 \end{pmatrix}$$

chirality projectors: $P_{R/L} = (\mathbb{1} \pm \gamma_5)/2 \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

$\Rightarrow$ for massless fermions $\quad u_p^+ = P_R u_p^+ \quad$ i.e. right chirality
$\qquad u_p^- = P_L u_p^- \quad$ left "

final anti fermion

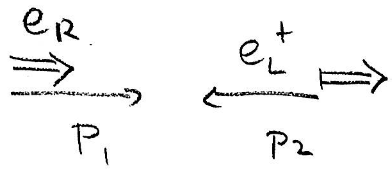
$\mu_R^+$

helicity of anti particle is opposite of spinor

$$v_p^{-} = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^- \\ -\sqrt{p\cdot\bar\sigma}\,\xi^- \end{pmatrix} = \begin{pmatrix} \sqrt{E+p}\,\xi^- \\ -\sqrt{E-p}\,\xi^- \end{pmatrix} \overset{m=0}{=} \sqrt{2E}\begin{pmatrix} \xi^- \\ 0 \end{pmatrix}$$

$$v_p^- = P_L\, v_p^-$$

initial state:




$$\begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}$$

$$\qquad e_L^+ \qquad e_R^-$$

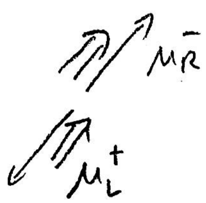
$$J^\mu_{ini} = \bar v_{\tilde p}\,\gamma^\mu\, u_p^+ = (\sqrt{2E})^2 \begin{pmatrix} 0 \\ \xi^- \end{pmatrix}^\dagger \gamma^0\gamma^\mu \begin{pmatrix} 0 \\ \xi^+ \end{pmatrix} = -E_{CM}\,\xi^-\sigma^\mu\xi^+$$

$\tilde p = (E\;0\;0\;-E)$

$\Rightarrow \vec{\tilde p}\cdot\vec\sigma = -\vec p\cdot\vec\sigma$

$$= -E_{cM}\,(0\,1)\,\sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -E_{cM}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$$

final state:




rotate initial by $\theta$: $\quad R(\theta)^\mu{}_\nu\, \bar v\gamma^\nu u = -R(\theta)\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} E_{CM}$

$$= -E_{CM}\begin{pmatrix} 0 \\ \cos\theta \\ i \\ -\sin\theta \end{pmatrix}$$

$$J^\mu_{final} = -E_{CM}(0\;\cos\theta\;-i\;-\sin\theta)$$

$$J^\mu_{ini}\cdot J_\mu^{final} = -E_{CM}^2(1+\cos\theta) = 2u$$

$$\mathcal{M} = -e^2\frac{2u}{s} \quad \checkmark$$