$$ {}_{\text{out}}\langle \vec{k}_1 \cdots \vec{k}_n \,|\, \vec{p}_1\, \vec{p}_2 \rangle_{\text{in}} $$

$$ = (2\pi)^4 \, \delta^4 \underbrace{(p_1 + p_2 - \textstyle\sum k_i)}_{\text{energy-mom. conservation}} \, \mathcal{M}(k_i, p_i) $$

Lorentz invariant matrix element, computed with Feynman rules

cross section

$$ \sigma = \frac{\#\ \text{scatters} / \text{time}}{\text{incident flux}} $$

$\sigma$ is a probability ~ square of scattering amplitude

$$ \sigma(p_1 p_2 \to k_1 \ldots k_n) = $$

4 momentum conserved due to $\delta^4$

$$ \underbrace{\frac{1}{2E_1 2E_2}}_{\substack{\text{rel. norm} \\ \text{of } |\vec{p}_1 \vec{p}_2\rangle}} \underbrace{\frac{1}{|v_1 - v_2|}}_{\text{from flux}} \underbrace{\prod_{i=1}^{n} \left( \int \frac{d^3 k_i}{(2\pi)^3 2E_i} \right)}_{\substack{\text{Lorentz invariant} \\ \text{"phase space" - integral over final momenta}}} (2\pi)^4 \delta^4 \underbrace{(p_1 + p_2 - \textstyle\sum_j k_j)}_{\text{E-mom cons.}} \underbrace{\left| \mathcal{M}(p_i, k_i) \right|^2}_{\substack{\text{squared matrix} \\ \text{element}}} $$

$\sigma$ is an area, invariant under boosts along beam axis (HW).

<u>Example</u>: 2 body phase space

$$\text{``}\int d\Pi_2\text{''} = \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} (2\pi)^4 \delta^4(\underbrace{p_1+p_2}-k_1-k_2)$$

$$E_{k_i} = \sqrt{\vec{k}_i^{\,2}+m_i^2}$$

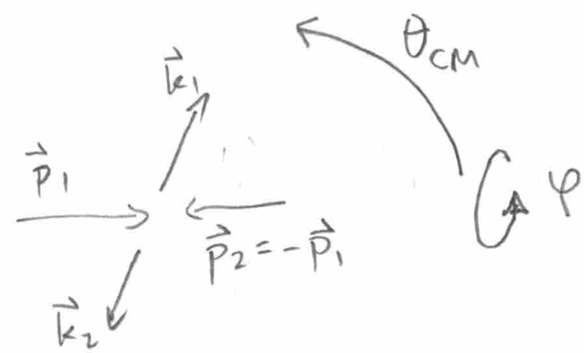
sum of 4 momenta of initial particles

do $\vec{k}_2$ integral with $\delta^3(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)$

$$= \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{1}{2E_{k_2}} 2\pi\, \delta(E_{p_1}+E_{p_2}-E_{k_1}-E_{k_2})$$

and $\vec{k}_2 = \vec{p}_1+\vec{p}_2-\vec{k}_1$

choose convenient frame: CM frame




$$\vec{p}_1 = -\vec{p}_2$$
$$\Rightarrow \vec{k}_2 = -\vec{k}_1$$
$$p_1+p_2 = (E_{p_1}+E_{p_2} \equiv E_{CM}, \vec{0})$$

$$\int d\Pi_2 = \int \frac{d\Omega}{(2\pi)^2} \int \frac{dk_1\, k_1^2}{2E_{k_1} 2E_{k_2}} \delta\left(E_{CM} - \sqrt{\vec{k}_1^{\,2}+m_1^2} - \sqrt{\vec{k}_2^{\,2}+m_2^2}\right)$$

assume $m_1 = m_2 \Rightarrow \delta\left(E_{CM} - 2\sqrt{k_1^2+m^2}\right)$

$$= \frac{1}{\left|\frac{2k_1}{\sqrt{k_1^2+m^2}}\right|}\,\delta\left(k_1 - \sqrt{\left(\frac{E_{CM}}{2}\right)^2 - m^2}\right)$$

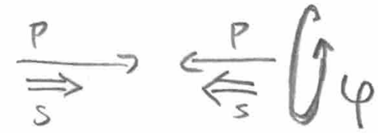
$$\int d\Pi_2 = \int \frac{d\Omega}{(2\pi)^2}\,\frac{k_1}{4E_{CM}}\Bigg|_{k_1 = \frac{E_{CM}^2}{4} - m^2}$$

$$= \frac{1}{32\pi^2}\int d\Omega\sqrt{1-\frac{(2m)^2}{E_{CM}^2}}$$

$$= \frac{1}{16\pi}\int_{-1}^{1} d(\cos\theta)\sqrt{1-\frac{(2m)^2}{E_{CM}^2}}$$

$$\int d\Omega = \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi}\int_0^{\pi}\underbrace{d\theta\sin\theta}_{\equiv d\cos\theta}$$

cylindrical symmetry of $|M|^2$




Q: what if $E_{CM} < 2m$, imaginary cross section?

A: $\delta\left(E_{CM} - 2\sqrt{k_1^2+m^2}\right) = 0$ there. $\Rightarrow$ below threshold for production of 2 particles of mass $m \Rightarrow \int d\Pi_2 = 0$

plugging back into $\sigma$:

$$\frac{d\sigma}{d\cos\theta} = \underbrace{\frac{1}{E_{CM}^2}\,\frac{1}{2}\,\frac{1}{16\pi}\sqrt{1-\frac{4m^2}{E_{CM}^2}}}_{\text{kinematics}}\ \underbrace{|M|^2}_{\text{physics}}$$

have our cross section master formula

$$\frac{d\sigma_{2\to 2}}{d\cos\theta} = \frac{1}{32\pi}\,\frac{1}{E_{CM}^2}\,\sqrt{1-\left(\frac{2m}{E_{CM}}\right)^2}\,\left|\mathcal{M}\right|^2$$

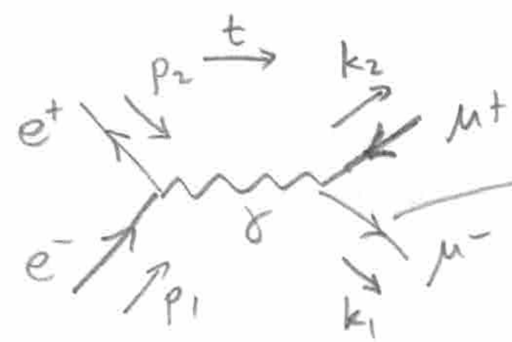
massive particles in final state          Feynman rules

example: $e^+e^- \to \mu^+\mu^-$

use helicity basis for spins $\Rightarrow$ $e^-_{L/R}$ for $\pm\frac{1}{2}$ helicity

$\begin{pmatrix} L \leftrightarrow - \\ R \leftrightarrow + \end{pmatrix}$

consider first $e^-_R e^+_L \to \mu^-_R \mu^+_L$     (note: $e^-_R \overset{CPT}{\longleftrightarrow} e^+_L$ )

FD :



arrows on fermion lines indicate fermion number flow. i.e. with time going to the right

$\Rightarrow$ $\longrightarrow$ fermion (e.g. $e^-$)

$\longleftarrow$ antifermion ($e^+$)

 stands for a formula for $\mathcal{M}$

let's guess it without Feynman rules ...

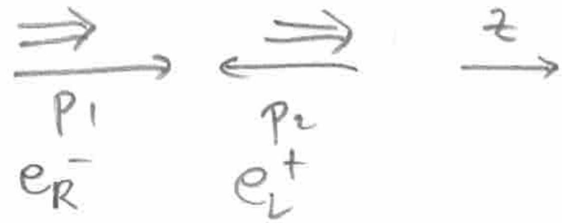
$\mathcal{M} \propto e^2$ from the two vertices 

in our units: $\alpha_{em} = \frac{e^2}{4\pi} = \frac{1}{137} \Rightarrow e \sim \frac{1}{3}$
($\hbar = c = 1$)

(S.I. units $\alpha_{em} = \frac{e_{SI}^2}{\hbar c}$)

spin dependence of $\mathcal{M}$ ?

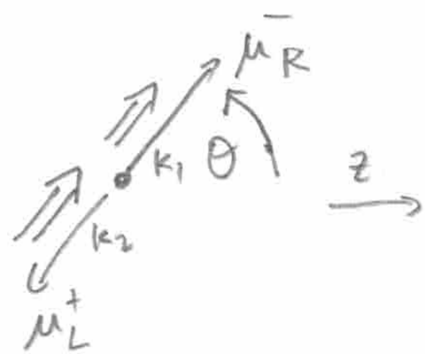
<u>initial state:</u>




$\Rightarrow$ combined spin of initial state $S_z = 1 \Rightarrow S = 1$

polarization vector of initial spin: $\vec{\epsilon}_+^{\,in} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

<u>final state:</u>




get final polarization vector by rotation (see lecture from last week)

$$\vec{\epsilon}_+^{\,out} = R_\theta \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_\theta \\ i \\ -s_\theta \end{pmatrix}$$

amplitude for $\Longrightarrow$ to be observed as $\nearrow$ ?

$$\mathcal{M} \propto e^2\, \vec{\epsilon}_+^{\,out\,\dagger} \cdot \vec{\epsilon}_+^{\,in} = e^2 \underline{(1 + c_\theta)}$$
$$2$$