Dynamics of Voter Models on Heterogeneous Networks

Sid Redner (physics.bu.edu/~redner)
Complex Network Program SAMSI Aug. 29-Sept. 1, 2010

T. Antal (Edinburgh), V. Sood (NBI)
NSF DMR0535503 & DMR0906504
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The classic voter model
3 basic results

Voting on complex networks
new conservation law & fixation probabilities
two time-scale route to consensus
short consensus time

Partisan voting
can truth be reached?

N. Gibert (Paris)
N. Masuda (Tokyo)
Classic Voter Model
Clifford & Sudbury (1973)
Holley & Liggett (1975)
0. Binary voter variable at each site $i$
0. Binary voter variable at each site $i$

1. Pick a random voter
Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   *individual has no self-confidence & adopts neighbor’s state*
0. Binary voter variable at each site $i$

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   *individual has no self-confidence & adopts neighbor’s state*
Example update:

0. Binary voter variable at each site $i$

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Example update:

0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor

*individual has no self-confidence & adopts neighbor’s state*

**proportional rule**

---

**Classic Voter Model**

Clifford & Sudbury (1973)
Holley & Liggett (1975)
0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   *individual has no self-confidence & adopts neighbor’s state*
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system
Voter Model Evolution

Dornic et al. (2001)

random initial condition, 256 x 256 square:

\[
\begin{array}{cccc}
\text{t=4} & \text{t=16} & \text{t=64} & \text{t=256} \\
\end{array}
\]
Voter Model Evolution

Dornic et al. (2001)

random initial condition, 256 x 256 square:

no surface tension
Voter Model & Cousins

Voter Model: Tell me how to vote

lemming
Voter Model & Cousins

**Voter Model:** Tell me how to vote

**Invasion Process:** I tell you how to vote
Voter Model & Cousins

**Voter Model:**
Tell me how to vote

**Invasion Process:**
I tell you how to vote

**Link Dynamics:**
Pick two disagreeing agents and change one at random
Voter Model & Cousins

Voter Model: Tell me how to vote

Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random

*identical on regular lattices, distinct on random graphs*

Lattice Voter Model: 3 Basic Properties
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) \)

Evolution of a single active link:
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

average magnetization conserved
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations

\[
\frac{\partial c_2(r, t)}{\partial t} = \nabla^2 c_2(r, t)
\]

\( c_2(r = 0, t) = 1 \)
\( c_2(r > 0, t=0) = 0 \)
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link:

2. Two-Spin Correlations

\[
\frac{\partial c_2(r, t)}{\partial t} = \nabla^2 c_2(r, t)
\]

\[
c_2(r > 0, t = 0) = 0
\]

- late time
- intermediate
- early

\[
c_2(r = 0, t) = 1
\]

\[
d > 2
\]
\[
d < 2
\]

\[
1 - (a/r)^{d-2}
\]
Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

   Evolution of a single active link:

   - Average magnetization conserved
   - \( \frac{1}{2} \) late time
   - \( \frac{1}{2} \) early

2. Two-Spin Correlations

\[
\frac{\partial c_2(r, t)}{\partial t} = \nabla^2 c_2(r, t)
\]

- \( c_2(r = 0, t) = 1 \)
- \( c_2(r > 0, t = 0) = 0 \)

\begin{align*}
   c_2(r > 0, t) &= 1 - (a/r)^{d-2} \\
   c_2(r = 0, t) &= 1 \\
   c_2(r > 0, t = 0) &= 0
\end{align*}

3. Consensus Time

\[
\int \sqrt{Dt} c(r, t) r^{d-1} dr = N
\]

<table>
<thead>
<tr>
<th>dimension</th>
<th>1</th>
<th>2</th>
<th>( &gt;2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>consensus time</td>
<td>( N^2 )</td>
<td>( N \ln N )</td>
<td>( N )</td>
</tr>
</tbody>
</table>
Voter Model on Complex Networks

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)
Voter Model on Complex Networks

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)

illustrative example: complete bipartite graph
Voter Model on Complex Networks

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)

illustrative example: complete bipartite graph

rate equation

\[ dN_{↑,a} = \frac{N_{↓,a} N_{↑,b} - N_{↑,a} N_{↓,b}}{(a + b)b} \]

\[ dN_{↑,b} = \frac{N_{↓,b} N_{↑,a} - N_{↑,b} N_{↓,a}}{(a + b)a} \]
**Voter Model on Complex Networks**

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL 69, 228 (2005)

**illustrative example: complete bipartite graph**

![Complete Bipartite Graph](image)

**rate equation**

\[
\begin{align*}
\frac{dN_{\uparrow,a}}{dt} &= \frac{N_{\downarrow,a}N_{\uparrow,b} - N_{\uparrow,a}N_{\downarrow,b}}{(a + b)b} \\
\frac{dN_{\uparrow,b}}{dt} &= \frac{N_{\downarrow,b}N_{\uparrow,a} - N_{\uparrow,b}N_{\downarrow,a}}{(a + b)a}
\end{align*}
\]

**Subgraph densities:**

\[
\rho_a = \frac{N_{\uparrow,a}}{a}, \quad \rho_b = \frac{N_{\uparrow,b}}{b} \quad dt = 1/(a + b)
\]

\[
\rho_{a,b}(t) = \frac{1}{2} \left[ \rho_{a,b}(0) - \rho_{b,a}(0) \right] e^{-2t} + \frac{1}{2} \left[ \rho_a(0) + \rho_b(0) \right]
\]

\[
\rightarrow \quad \frac{1}{2} \left[ \rho_a(0) + \rho_b(0) \right]
\]

**magnetization not conserved**
Voter Model on Complex Networks
Voter Model on Complex Networks

low degree; picked often

import often
Voter Model on Complex Networks

- High degree; few nodes change rarely
- Low degree; picked often import often
Voter Model on Complex Networks

- High degree; few nodes change rarely
- Low degree; picked often imported often

"flow" from high degree to low degree
Invasion Process on Complex Networks

Castellano, AIP Conf Proc 779, 114 (2005)
Invasion Process on Complex Networks

Castellano, AIP Conf Proc 779, 114 (2005)

high degree; change often

low degree; picked often export often
Invasion Process on Complex Networks

Castellano, AIP Conf Proc 779, 114 (2005)

“flow” from low degree to high degree
Formal Approach for Conservation Law

flip rate: \[ P[\eta \rightarrow \eta_x] = \sum_{y} \frac{A_{xy}}{Z} \left[ \Phi(x, y) + \Phi(y, x) \right] \]

\[ \eta = \{1, 1, 0, 0, \ldots, 1\} \quad \text{system state} \]

\[ \eta_x = \text{system state when voter at } x \text{ flips} \]

\[ \eta(x) = \text{state of voter at } x \]
Formal Approach for Conservation Law

$\Phi(x, y) \equiv \eta(x)[1 - \eta(y)]$

$A_{xy} = \text{adjacency matrix}$

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Formal Approach for Conservation Law

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2 connected nodes in different states

flip rate: \[ P[\eta \rightarrow \eta_x] = \sum_y \frac{A_{xy}}{2} [\Phi(x, y) + \Phi(y, x)] \]

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\[ \eta_x = \text{system state when voter at } x \text{ flips} \]
\[ \eta(x) = \text{state of voter at } x \]

\[ Z \equiv \begin{cases} 
Nk_x & \text{VM} \\
Nk_y & \text{IP} \\
N\mu_1 & \text{LD} 
\end{cases} \]

Choose \( x \), choose neighbor of \( x \) with prob. \( (Nk_x)^{-1} \)

Choose \( y \) (neighbor of \( x \)), choose of \( x \) with prob. \( (Nk_y)^{-1} \)

Choose link & update \( x \) with prob. \( (N\mu_1)^{-1} \)

**2 connected nodes in different states**
Formal Approach for Conservation Law

\[ \langle \Delta \eta(x) \rangle = [1 - 2\eta(x)] P[\eta \rightarrow \eta_x] = \sum_y \frac{A_{xy}}{Z} [\eta(y) - \eta(x)] \]
Formal Approach for Conservation Law

\[ \langle \Delta \eta(x) \rangle = [1 - 2\eta(x)] \mathbf{P}[\eta \rightarrow \eta_x] = \sum_y \frac{A_{xy}}{Z} [\eta(y) - \eta(x)] \]

degree-weighted moments

\[ \langle \omega_m \rangle \equiv \frac{1}{N \mu_m} \sum_x k_x^m \eta(x) = \frac{1}{\mu_m} \sum_k k^m n_k \rho_k \]
Formal Approach for Conservation Law

\[ \langle \Delta \eta(x) \rangle = [1 - 2\eta(x)] \mathbf{P}[\eta \rightarrow \eta_x] = \sum_y \frac{A_{xy}}{Z} [\eta(y) - \eta(x)] \]

degree-weighted moments

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\[ \Delta \langle \omega_1 \rangle = \sum_{x,y} \frac{A_{xy}}{N k_x} k_y [\eta(y) - \eta(x)] = 0 \]

voter model

\[ \Delta \langle \omega_{-1} \rangle = \sum_{x,y} \frac{A_{xy}}{N k_x k_y} [\eta(y) - \eta(x)] = 0 \]

invasion process

\[ \langle \Delta \omega_0 \rangle = \langle \Delta \rho \rangle = \sum_{x,y} \frac{A_{xy}}{N \mu_1} [\eta(y) - \eta(x)] = 0 \]

link dynamics
Exit Probability on Complex Networks

Voter model: \( \mathcal{E}(\omega) = \omega \)
Exit Probability on Complex Networks

Voter model: \( E(\omega) = \omega \)

Extreme case: star graph

N nodes: degree 1
1 node: degree N
Exit Probability on Complex Networks

Voter model: $\mathcal{E}(\omega) = \omega$

Extreme case: star graph

N nodes: degree 1
1 node: degree N

$\omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k = \frac{1}{2}$

Final state: all 1 with prob. 1/2!
Byproduct: Voter Model Fixation Probability

What is the probability that a single mutant “takes over” a population?
Byproduct: Voter Model Fixation Probability

What is the probability that a single mutant “takes over” a population?

fixation probability $\propto$ node degree

1 consensus
Invasion Process Fixation Probability

$\text{fixation probability } \propto \frac{1}{(\text{node degree})}$

0 consensus
Route to Consensus on Complex Graphs

complete bipartite graph

\[ t \lesssim 1 \]
Route to Consensus on Complex Graphs

complete bipartite graph

a sites
degree b

b sites
degree a

t \lesssim 1

N=10000, C links/node

two-clique graph

c = 100

c = 1

ρ
b

ρ
a

ρ
b
Consensus Time Evolution Equation

A Guide to First-Passage Processes
(CUP, 2001)

warmup: complete graph

\( T(\rho) \equiv \text{av. consensus time starting with density } \rho \)

\[
T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
+ \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
+[1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\]
Consensus Time Evolution Equation

*complete graph*

\[ T(\rho) \equiv \text{av. consensus time starting with density } \rho \]

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T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
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+ [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
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\[ \mathcal{R}(\rho) \equiv \text{prob}(\downarrow \uparrow \rightarrow \uparrow \uparrow) = \rho(1 - \rho) \]
Consensus Time Evolution Equation

warmup: complete graph

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Consensus Time Evolution Equation

warmup: complete graph

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T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
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+ [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\]

\[
\mathcal{R}(\rho) \equiv \text{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\
\mathcal{L}(\rho) \equiv \text{prob}(\uparrow\downarrow\rightarrow\downarrow\downarrow) \\
= \rho(1 - \rho)
\]
Consensus Time on Complete Graph

\[ T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \]
\[ + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \]
\[ + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt] \]

continuum limit:
\[ T'' = -\frac{N}{\rho(1 - \rho)} \]

solution:
\[ T(\rho) = -N \left[ \rho \ln \rho + (1 - \rho) \ln(1 - \rho) \right] \]
Consensus Time on Heterogeneous Networks

\[
T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k \\
\text{on nodes of degree } k
\]

\[
T(\{\rho_k\}) = \sum_k R_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt] \\
+ \sum_k L_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt] \\
+ \left[1 - \sum_k [R_k(\{\rho_k\}) + L_k(\{\rho_k\})]\right][T(\{\rho_k\}) + dt]
\]

\[
R_k(\{\rho_k\}) = \text{prob}(\rho_k \rightarrow \rho_k^+) \\
= \frac{1}{N} \sum_x \frac{1}{k_x} \sum_y P(\downarrow, \overline{\quad}, \uparrow) \\
= n_k \omega (1 - \rho_k)
\]

\[
L_k(\{\rho_k\}) = n_k \rho_k (1 - \omega)
\]
Consensus Time on Heterogeneous Networks

continuum limit:

\[
\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1
\]
(Molloy-Reed) Configuration Model

\[ n_k \sim k^{-2.5}, \quad \mu_1 = 8 \]
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2N n_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $\rho_k \to \omega \forall k$

and

$$\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{k n_k}{\mu_1} \frac{\partial}{\partial \omega}$$
Consensus Time on Heterogeneous Networks

continuum limit:

\[
\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1
\]

now use \( \rho_k \to \omega \) \( \forall k \)

and

\[
\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega}
\]

to give

\[
\frac{\partial^2 T}{\partial \omega^2} = -\frac{N \mu_1^2 / \mu_2}{\omega(1 - \omega)}
\]

same as \( T'' = -\frac{N}{\rho(1 - \rho)} \)

with effective size \( N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \)
Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

Voter model: $T_N \sim N_{\text{eff}} = N \mu_1^2/\mu_2$

$T_N \sim \begin{cases} 
N & \nu > 3, \\
N/\ln N & \nu = 3, \\
N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\
(\ln N)^2 & \nu = 2, \\
\mathcal{O}(1) & \nu < 2.
\end{cases}$
Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

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\mathcal{O}(1) & \nu < 2.
\end{cases}$$

Invasion process: $T_N \sim N_{\text{eff}} = N \mu_1 \mu_{-1}$

$$T_N \sim \begin{cases} 
N & \nu > 2, \\
N \ln N & \nu = 2, \\
N^{3-\nu} & \nu < 2.
\end{cases}$$
Partisan Voting and Truth

N. Masuda, N. Gibert, SR
PRE 82, 010103(R) (2010)

\[ \begin{align*}
  &\text{prefers truth} & \text{& in T state} & \text{density } T_+ \\
  &\text{prefers truth} & \text{& in F state} & \text{density } T_- \\
  &\text{prefers false} & \text{& in T state} & \text{density } F_+ \\
  &\text{prefers false} & \text{& in F state} & \text{density } F_- 
\end{align*} \]
Partisan Voting and Truth

N. Masuda, N. Gibert, SR
PRE 82, 010103(R) (2010)

prefers truth & in T state
density $T_+$

prefers truth & in F state
density $T_-$

prefers false & in T state
density $F_+$

prefers false & in F state
density $F_-$

fraction that believe the truth
partisan voting update:

1. Pick voter, pick neighbor (as in usual voter model);

2a. If initial voter becomes *concordant* by adopting neighboring state, change occurs with rate $1+\varepsilon$;

2b. If initial voter becomes *discordant* by adopting neighboring state, change occurs with rate $1-\varepsilon$. 
Rate Equations

\[
\begin{align*}
\dot{T}_+ &= (1 + \epsilon)T_- [T_+ + F_-] - (1 - \epsilon)T_+ [T_- + F_+] \\
\dot{T}_- &= (1 - \epsilon)T_+ [T_- + F_+] - (1 + \epsilon)T_- [T_+ + F_-] \\
\dot{F}_+ &= (1 + \epsilon)F_- [F_+ + T_-] - (1 - \epsilon)F_+ [F_- + T_+] \\
\dot{F}_- &= (1 - \epsilon)F_+ [F_- + T_+] - (1 + \epsilon)F_- [F_+ + T_-] \\
\end{align*}
\]

\[
\begin{align*}
T_- &= T - T_+ & S &= T_+ + F_+ \\
F_- &= F - F_+ & \Delta &= T_+ - F_+ \\
\end{align*}
\]
\[ S = T_+ + F_+ \]
\[ \Delta = T_+ - F_+ \]

Flow Diagram

\[ \epsilon < 2T - 1 \]
\[ \epsilon > 2T - 1 \]
Summary & Outlook

Voter model:
paradigmatic, soluble, (but hopelessly naive)

Voter model on complex networks:
new conservation law
two time-scale route to consensus
fast consensus for broad degree distributions

Extension to Partisanship:
partisanship forestalls consensus to the truth

Future:
“churn” rather than consensus
heterogeneity of real people
positive and negative social interactions
Aimed at graduate students, this book explores some of the core phenomena in non-equilibrium statistical physics. It focuses on the development and application of theoretical methods to help students develop their problem-solving skills.

The book begins with microscopic transport processes: diffusion, collision-driven phenomena, and exclusion. It then presents the kinetics of aggregation, fragmentation, and adsorption, where basic phenomenology and solution techniques are emphasized. The following chapters cover kinetic spin systems, by developing both a discrete and a continuum formulation, the role of disorder in non-equilibrium processes, and hysteresis from the non-equilibrium perspective. The concluding chapters address population dynamics, chemical reactions, and a kinetic perspective on complex networks.

The book contains more than 200 exercises to test students’ understanding of the subject. A link to a website hosted by the authors, containing an up-to-date list of errata and solutions to some of the exercises, can be found at www.cambridge.org/9780521851039.

Pavel L. Krapivsky is Research Associate Professor of Physics at Boston University. His current research interests are in strongly interacting many-particle systems and their applications to kinetic spin systems, networks, and biological phenomena.

Sidney Redner is a Professor of Physics at Boston University. His current research interests are in non-equilibrium statistical physics and its applications to reactions, networks, social systems, biological phenomena, and first-passage processes.

Eli Ben-Naim is a member of the Theoretical Division and an affiliate of the Center for Nonlinear Studies at Los Alamos National Laboratory. He conducts research in statistical, nonlinear, and soft condensed-matter physics, including the collective dynamics of interacting particle and granular systems.

Cover illustration: Snapshot of a collision cascade in a perfectly elastic, initially stationary hard-sphere gas in two dimensions due to a single incident particle. Shown are the cloud of moving particles (red) and the stationary particles (blue) that have not yet experienced any collisions. Figure courtesy of Tibor Antal.