The Dynamics of Persuasion
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CIRM, Luminy France, January 5-9, 2015
+ support from NSF

Modeling Consensus:
• introduction to the voter model
• voter model on complex networks
• voting with some confidence
• majority rule

Modeling Discord & Diversity:
• 3-state voter models
• strategic voting
• bounded compromise
• dynamics of social balance
• Axelrod model

lecture 1
lecture 2
lecture 3
Persuasion Dynamics

Real People

People as interacting “atoms”
Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   *individual has no self-confidence & adopts neighbor’s state*
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   \textit{individual has no self-confidence \& adopts neighbor's state}
Voter Model

Example update:

0. Binary voter variable at each site $i$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
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Clifford & Sudbury (1973)
Holley & Liggett (1975)
Voter Model

Example update:

0. Binary voter variable at each site i

1. Pick a random voter

2. Assume state of randomly-selected neighbor individual has no self-confidence & adopts neighbor’s state

3. Repeat 1 & 2 until consensus necessarily occurs in a finite system

Clifford & Sudbury (1973)
Holley & Liggett (1975)
**Voter vs. Ising Models**

**Voter model:** *proportional rule*
- Clifford & Sudbury (1973)
- Holley & Liggett (1975)
- Consensus *inevitable* in a finite system

**Kinetic Ising model:** *majority rule at T=0*
- Glauber (1963)
- Consensus *not inevitable* in a finite system
Voter Evolution vs. Ising Evolution

Voter
Dornic et al. (2001)

random initial condition:
droplet initial condition:

random initial condition:
droplet initial condition:
Lattice Voter Model: 3 Basic Properties
1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Evolution of a single active link (homogeneous network):
I. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

Evolution of a single active link (homogeneous network):
1. Final State (Exit) Probability \( \mathcal{E}(\rho_0) = \rho_0 \)

Evolution of a single active link (homogeneous network):

average magnetization is conserved!
2. Spatial Dependence of 2-Spin Correlations

flip rate:

\[ w_i = \frac{1}{2} \left[ 1 - \frac{\sigma_i}{z} \sum_{j \in \langle i \rangle} \sigma_j \right] \]

1-spin correlations:

\[ \frac{d\langle \sigma_i \rangle}{dt} = -2\langle \sigma_i w_i \rangle \]
2. Spatial Dependence of 2-Spin Correlations

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1-spin correlations:
\[ \frac{d\langle \sigma_i \rangle}{dt} = -2\langle \sigma_i w_i \rangle \]
\[ = -\langle \sigma_i \rangle + \frac{1}{z} \sum_j \langle \sigma_j \rangle \]

\[ \langle \sigma_i(t) \rangle = I_i(t) e^{-t} \]
for \( \langle \sigma_i(t=0) \rangle = \delta_{i,0} \)

2-spin correlations:
\[ \frac{d\langle \sigma_i \sigma_j \rangle}{dt} = -2\langle \sigma_i \sigma_j (w_i + w_j) \rangle \]
\[ = -2\langle \sigma_i \sigma_j \rangle + \frac{1}{2d} \left( \sum_{k \in \langle i \rangle} \langle \sigma_k \sigma_j \rangle + \sum_{k \in \langle j \rangle} \langle \sigma_i \sigma_k \rangle \right) \]
2. Spatial Dependence of 2-Spin Correlations

(\text{infinite system})

\text{Equation for 2-spin correlation function:}

\[
\frac{d\langle \sigma_i \sigma_j \rangle}{dt} = -2\langle \sigma_i \sigma_j (w_i + w_j) \rangle
\]

\[
\frac{\partial c_2(r, t)}{\partial t} = \nabla^2 c_2(r, t)
\]

\[c(r = 0, t) = 1; \quad c(r > 0, t = 0) = 0\]
2. Spatial Dependence of 2-Spin Correlations

(infinite system)

Equation for 2-spin correlation function:

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Equation for 2-spin correlation function:

\[
\frac{d \langle \sigma_i \sigma_j \rangle}{dt} = -2 \langle \sigma_i \sigma_j (w_i + w_j) \rangle
\]

Asymptotic solution:

\[
c(r, t) \sim \begin{cases} 
1 - \frac{1 - (\frac{a}{r})^{d-2}}{1 - (\frac{a}{\sqrt{D}r})^{d-2}} & \text{if } d \neq 2 \\
1 - \frac{\ln r}{\ln a} & \text{if } d = 2
\end{cases}
\]

\[
d \neq 2 \\
d = 2
\]

Spatial Dependence of 2-Spin Correlations

\[
c(r=0, t) = 1; \quad c(r > 0, t = 0) = 0
\]
3. System Size Dependence of Consensus Time

Liggett (1985), Krapivsky (1992)

\[
\int \sqrt{D t} \ c(r, t) r^{d-1} \ dr = N
\]

<table>
<thead>
<tr>
<th>dimension</th>
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</tr>
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<tbody>
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<td>1</td>
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<td>$N$</td>
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Lattice Voter Model: 3 Basic Properties

1. Final State Probability

\[ \mathcal{E}(\rho_0) = \rho_0 \]

Evolution of a single active link:

average magnetization conserved

2. Two-Spin Correlations

\[ \frac{\partial c(r, t)}{\partial t} = \nabla^2 c(r, t) \]

\[ c(r = 0, t = 0) = 1 \]

\[ c(r > 0, t = 0) = 0 \]

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Voter Model on Complex Networks

magnetization on regular networks

magnetization not conserved on complex networks

“flow” from high degree to low degree

Suchecki, Eguiluz & San Miguel (2005)
New Conservation Law

Sood & SR (2005)

to compensate the different rates:

degree-weighted 1st moment:

\[ \omega \equiv \frac{\sum_k kn_k \rho_k}{\sum_k kn_k} = \frac{\sum_k kn_k \rho_k}{\mu_1} \]

\[ \mu_1 = \text{av. degree} \]

\[ n_k = \text{frac. nodes of degree } k \]

\[ \rho_k = \text{frac. } \uparrow \text{ on nodes of degree } k \]
Conservation Law for Voter Model

Transition probability

\[ P[\eta \rightarrow \eta_x] = \sum_y \frac{A_{xy}}{Nk_x} \left[ \Phi(x, y) + \Phi(y, x) \right] \]

\( \eta \equiv \) state of system
\( \eta_x \equiv \) state after flip at \( x \)
\( \eta(x) \equiv \) state at \( x \) (0,1)

Density change:

\[ \langle \Delta \eta(x) \rangle = [1 - 2\eta(x)] P[\eta \rightarrow \eta_x] \]

Degree-weighted moments:

\[ \omega_n = \frac{1}{N \mu_n} \sum_x k_x^n \eta(x) \]

Change in weighted first moment:

\[ \langle \Delta \omega_1 \rangle = \sum_{x,y} \frac{A_{xy}}{Nk_x} k_x [\eta(y) - \eta(x)] = 0 \] conserved!
Exit Probability on Complex Graphs

\[ \mathcal{E}(\omega) = \omega \]

Extreme case: star graph

\[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

N nodes: degree 1
1 node: degree N

\[ \omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k = \frac{1}{2} \]

Final state: all 1 with prob. 1/2!
Voter Model on Complex Networks

complete bipartite graph

- a sites
- degree b

b sites
- degree a

t \lesssim 1

\( \rho_a, \rho_b \)

\( N = 10000, C \text{ links/node} \)

two-clique graph

- a sites
- degree b

\( c = 1 \)

\( c = 100 \)

\( \sim \)

\( \rho_a, \rho_b \)

Sucheki, Eguiluz & San Miguel (2005)
Sood & SR (2005)
Antal, Sood & SR (2005)
Consensus Time Evolution Equation

warmup: complete graph

\[ T(\rho) \equiv \text{av. consensus time starting with density } \rho \]

\[
T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\
+ \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\
+ [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\]
**Consensus Time Evolution Equation**

**warmup: complete graph**

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\]

\[ \mathcal{R}(\rho) \equiv \text{prob}(\downarrow \uparrow \rightarrow \uparrow \uparrow) = \rho(1 - \rho) \]
Consensus Time Evolution Equation

warmup: complete graph

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\[ = \rho(1 - \rho) \]
Consensus Time Evolution Equation

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\[ \mathcal{R}(\rho) \equiv \text{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \]
\[ \mathcal{L}(\rho) \equiv \text{prob}(\uparrow\downarrow\rightarrow\downarrow\downarrow) \]
\[ = \rho(1 - \rho) \]
Consensus Time on Complete Graph

\[ T(\rho) = \mathcal{R}(\rho)[T(\rho + d\rho) + dt] + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt] \]

continuum limit: \[ T'' = -\frac{N}{\rho(1 - \rho)} \]

solution: \[ T(\rho) = -N \left[ \rho \ln \rho + (1 - \rho) \ln(1 - \rho) \right] \]
Consensus Time on Heterogeneous Networks

\[ T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k \text{ on nodes of degree } k \]

\[ T(\{\rho_k\}) = \sum_k R_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt] \]

\[ + \sum_k L_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt] \]

\[ + \left[ 1 - \sum_k [R_k(\{\rho_k\}) + L_k(\{\rho_k\})] \right][T(\{\rho_k\}) + dt] \]

\[ R_k(\{\rho_k\}) = \text{prob}(\rho_k \rightarrow \rho_k^+) \]

\[ = \frac{1}{N} \sum'_x \frac{1}{k_x} \sum_y P(\downarrow, \rightarrow, \uparrow) \]

\[ = n_k \omega (1 - \rho_k) \]

\[ L_k(\{\rho_k\}) = n_k \rho_k (1 - \omega) \]
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2N n_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$
Voter Model on Complex Networks

configuration model

\[ n_k \sim k^{-2.5}, \quad \mu_1 = 8 \]
Consensus Time on Heterogeneous Networks

continuum limit:

\[
\sum_k \left[ (\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega \rho_k}{2N n_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1
\]

now use \( \rho_k \to \omega \quad \forall k \)

and

\[
\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{k n_k}{\mu_1} \frac{\partial}{\partial \omega}
\]

to give

\[
\frac{\partial^2 T}{\partial \omega^2} = -\frac{N \mu_1^2 / \mu_2}{\omega(1 - \omega)} \quad \text{same as} \quad T'' = -\frac{N}{\rho(1 - \rho)}
\]

with effective size \( N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \)

\[
\mu_n = \sum_k k^n n_k
\]
Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2}$$
Consensus Time for Complex Networks

with \( n_k \sim k^{-\nu} \)

\[
T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \sim \begin{cases} 
N & \nu > 3 \\
\nu > 3 
\end{cases}
\]
Consensus Time for Complex Networks with $n_k \sim k^{-\nu}$

$$T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \sim \begin{cases} 
N & \nu > 3 \\
N/\ln N & \nu = 3 
\end{cases}$$
Consensus Time for Complex Networks

with $n_k \sim k^{-\nu}$

$T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \sim \left\{ \begin{array}{ll}
N & \nu > 3 \\
N/\ln N & \nu = 3 \\
N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\
(\ln N)^2 & \nu = 2 \\
O(1) & \nu < 2
\end{array} \right.$

fast consensus

Invasion process:

$T_N \sim \left\{ \begin{array}{ll}
N & \nu > 2, \\
N \ln N & \nu = 2, \\
N^{2-\nu} & \nu < 2.
\end{array} \right.$
“Confident” Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)

unsure

confident
“Confident” Voter Model

classification: Centola (2010)
related work: Dall’Asta & Castellano (2007)

unsure

confident
“Confident” Voter Model

motivation: Centola (2010)
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“Confident” Voter Model

motivation: Centola (2010)
related work: Dall’Asta & Castellano (2007)
Simplest case: 2 internal states

densities $P_0, P_1, M_0, M_1$, with $P_0 + P_1 + M_0 + M_1 = 1$

basic processes:

\[
\begin{align*}
M_1 P_1 & \rightarrow P_0 P_1 \text{ or } M_0 M_1 \\
P_0 P_1 & \rightarrow P_0 P_1 \text{ or } P_0 P_0 \\
M_1 P_0 & \rightarrow M_1 P_1 \text{ or } P_0 P_0 \\
\end{align*}
\]

\[
\begin{align*}
M_0 P_0 & \rightarrow M_0 P_1 \text{ or } M_1 P_0 \\
M_0 M_1 & \rightarrow M_0 M_1 \text{ or } M_0 M_0 \\
M_0 P_1 & \rightarrow M_1 P_1 \text{ or } M_0 M_0 \\
\end{align*}
\]

rate equations/mean-field limit:

\[
\begin{align*}
\dot{P}_0 &= -M_0 P_0 + M_1 P_1 + P_0 P_1 \\
\dot{P}_1 &= M_0 P_0 - M_1 P_1 - P_0 P_1 + (M_1 P_0 - M_0 P_1)
\end{align*}
\]

similarly for $M_0, M_1$
special soluble case: symmetric limit

\[ P_0 + P_1 = M_0 + M_1 = \frac{1}{2} \]

\[ \dot{P}_0 = -M_0P_0 + M_1P_1 + P_0P_1 \]
\[ \dot{P}_1 = M_0P_0 - M_1P_1 - P_0P_1 + (M_1P_0 - M_0P_1) \]

\[ \rightarrow \quad \dot{P}_0 = -\dot{P}_1 = P_0^2 + \frac{1}{2}P_0 - \frac{1}{4} \]
\[ = -(P_0 - \lambda_+)(P_0 - \lambda_-) \]
\[ \lambda_\pm = \frac{1}{4}(-1 \pm \sqrt{5}) \approx 0.309, -0.809 \]

solution:

\[
\frac{P_0(t) - \lambda_+}{P_0(t) - \lambda_-} = \frac{P_0(0) - \lambda_+}{P_0(0) - \lambda_-} e^{-(\lambda_+ - \lambda_-)t}
\]
near symmetric limit:

\[ P_0 = \frac{1}{2} + 10^{-5}, \quad M_0 = \frac{1}{2} - 10^{-5}, \quad P_1 = M_1 = 0 \]
near symmetric limit: composition tetrahedron
Consensus Time in Two Dimensions

\[ T_N \]

\[ N \]

Log-log plot showing the relationship between \( T_N \) and \( N \).
Consensus Time Distribution

P(T_N)

T

droplets

stripes
two time scales control approach to consensus

see also Spirin, Krapivsky, SR (2001), Chen & SR (2005)

Ising model

Majority vote model
Majority rule

1. Pick a random group of $G$ spins (with $G$ odd).
2. All spins in $G$ adopt the majority state.
3. Repeat until consensus necessarily occurs.

Basic questions: 1. Which final state is reached? 2. What is the time until consensus?
Mean-field theory (for G=3)

\[ E_n \equiv \text{exit probability to } m = 1 \text{ starting from } n \text{ plus spins} \]

\[ = p_n E_{n+1} + q_n E_{n-1} + r_n E_n \]

where

\[ p_n = \binom{3}{2} \frac{(N-3)}{(n-2)} / \binom{N}{n} \]

\[ q_n = \binom{3}{1} \frac{(N-3)}{(n-1)} / \binom{N}{n} \]

\[ r_n = 1 - p_n - q_n \]

\[ T_n \equiv \text{mean time to } m = 1 \text{ starting from } n \text{ plus spins} \]

\[ = p_n (T_{n+1} + \delta t) + q_n (T_{n-1} + \delta t) + r_n (T_n + \delta t) \]
Consensus time for finite spatial dimensions

Critical dimension appears to be >4!
Anomalous dynamics in 2d: stripes $\sim 33\%$ of the time!
Slab formation in 3d ~8% of the time
Consensus time distribution

2d

3d

4d

 multiscale relaxation to final consensus
The Dynamics of Persuasion
Sid Redner, Santa Fe Institute (physics.bu.edu/~redner)
CIRM, Luminy France, January 5-9, 2015

+ support from NSF

Modeling Consensus:
• introduction to the voter model
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• voting with some confidence
• majority rule

Modeling Discord & Diversity:
• 3-state voter models
• strategic voting
• bounded compromise
• dynamics of social balance
• Axelrod model
Discord & Diversity

If people are reasonable, why is consensus hard to reach?

Possibilities:

- insufficient communication
- appreciable diversity
- stubbornness
- …. your favorite mechanism

Models:

Three voting states: — 0 +; — + noninteracting

Strategic voting: ideology vs. strategy

Bounded confidence: compromise only when close

Social balance: dynamics of positive/negative links

Axelrod model: many features, many traits

References:

- Deffuant, Neau, Amblard & Weisbuch (2000)
- Hegselmann & Krause (2002)
- Volovik, Mobilia & SR (2009)
- Axelrod (1997)
- Castellano, Marsili & Vespignani (2000)
- Vazquez & SR (2007)
Three Voting States: $- 0 +$

0. 3-state voter at each site: $- 0 +$
1. Pick a random voter
2. Assume state of neighbor if compatible
3. Repeat until either consensus or frozen final state

$N_0, N_- \rightarrow (N_0 \pm 1, N_- \mp 1)$ prob. $\frac{N_0 N_-}{N^2} = \rho_0 \rho_-$

$N_0, N_+ \rightarrow (N_0 \pm 1, N_+ \mp 1)$ prob. $\frac{N_0 N_+}{N^2} = \rho_0 \rho_+$

$+ - \rightarrow + -$ compatible

$+ - \rightarrow + -$ incompatible
Evolution in Composition Triangle

\[
\rho_0 = \Gamma_0 + 1 - 2 \rho_1
\]

\[
\Gamma_1 = \Gamma_0
\]

\[
\rho_\perp = \Gamma_0 + 1 - 2 \rho_1 = \Gamma_0 + 2 \rho_1
\]

consensus

frozen
Evolution in Composition Triangle

\[ \rho_0 \]

\[ \rho_- \]

\[ \rho_+ \]

\[ 1 - 2p_x - 2p_y \]

\[ p_x = \rho_0 \rho_- \]

\[ p_y = \rho_0 \rho_+ \]
The Phase Diagram

\( F(\rho_-, \rho_+) = \text{prob. to reach frozen state starting from } (\rho_-, \rho_+) \)

**recursion:**

\[
F(\rho_-, \rho_+) = p_x [F(\rho_- - \delta, \rho_+) + F(\rho_- + \delta, \rho_+)] \\
= p_y [F(\rho_, \rho_+ - \delta) + F(\rho_, \rho_+ + \delta)] \\
= [1 - 2(p_x + p_y)] F(\rho_-, \rho_+)
\]

**continuum limit:**

\[
\rho_- \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_-^2} + \rho_+ \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_+^2} = 0
\]

\[
F(\rho_-, \rho_+) = \sum_{n \text{ odd}} \frac{2(2n + 1)}{n(n + 1)} \sqrt{\rho_- \rho_+} (\rho_- + \rho_+)^n P_n^1 \left( \frac{\rho_- - \rho_+}{\rho_- + \rho_+} \right)
\]

\[
F(\rho_0) = 1 - \frac{1 - (1 - \rho_0)^2}{\sqrt{1 + (1 - \rho_0)^2}}
\]

**symmetric limit**
The Phase Diagram

\[ F(\rho_-, \rho_+) = \text{prob. to reach frozen state starting from } (\rho_-, \rho_+) \]

**Recursion:**
\[
F(\rho_-, \rho_+) = p_x [F(\rho_- - \delta, \rho_+) + F(\rho_- + \delta, \rho_+)] \\
= p_y [F(\rho_-, \rho_+ - \delta) + F(\rho_-, \rho_+ + \delta)] \\
= [1 - 2(p_x + p_y)] F(\rho_-, \rho_+)
\]

**Continuum Limit:**
\[
\rho_- \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_-^2} + \rho_+ \frac{\partial^2 F(\rho_-, \rho_+)}{\partial \rho_+^2} = 0
\]

Boundary Conditions:
\[
F(\rho_-, 0) = 0 \\
F(0, \rho_+) = 0 \\
F(\rho_+, 1-\rho_+) = 1
\]

Symmetric Limit:
\[
F(\rho_0) = 1 - \frac{1 - (1 - \rho_0)^2}{\sqrt{1 + (1 - \rho_0)^2}}
\]

In each region, prob. to reach specified state is > 50%
Phase Diagram & Final State Probabilities

in each region, prob. to reach specified state is > 50%

moral: extremism promotes deadlock
Strategic Voting

¿vote for first choice?
¿vote against last choice?

UK elections 1830-2010

Canadian elections 1935-1990
Strategic Voter Model

Evolution of the densities $a$, $b$, $c$:

$$\dot{a} = \frac{1}{T}(b + c^2 a) + r_{AC}ac + r_{AB}ab$$

$$\dot{b} = \frac{1}{T}(c + a^2 b) + r_{BA}ba + r_{BC}bc$$

$$\dot{c} = \frac{1}{T}(a + b^2 c) + r_{CA}ca + r_{CB}cb$$
Strategic Voter Model

 Evolution of the densities $a$, $b$, $c$:

\[
\begin{align*}
\dot{a} &= T(b + c - 2a) \\
\dot{b} &= T(c + a - 2b) \\
\dot{c} &= T(a + b - 2c)
\end{align*}
\]
Strategic Voter Model

evolution of the densities $a$, $b$, $c$:

$$
\dot{a} = T(b + c - 2a) + r_{AC} ac + r_{AB} ab
$$

$$
\dot{b} = T(c + a - 2b) + r_{BA} ba + r_{BC} bc
$$

$$
\dot{c} = T(a + b - 2c) + r_{CA} ca + r_{CB} cb
$$

- temperature
- strategic voting
Strategic Voter Model

evolution of the densities \(a, b, c\):

\[
\dot{a} = T(b + c - 2a) + r_{AC} ac + r_{AB} ab \\
\dot{b} = T(c + a - 2b) + r_{BA} ba + r_{BC} bc \\
\dot{c} = T(a + b - 2c) + r_{CA} ca + r_{CB} cb
\]

strategic voting rates:
\[
r_{AB} = -r_{BA} = \begin{cases} +r & B \text{ minority} \\ 0 & C \text{ minority} \\ -r & A \text{ minority} \end{cases}
\]

two natural choices
\[
r = \text{const.} \\
r = r_0[(a + b)/2 - c]
\]

\[
\dot{a} = T(1 - 3a) + rac \\
\dot{b} = T(1 - 3b) + rbc \quad \text{in } c< \text{ sector} \\
\dot{c} = T(1 - 3c) - rc(1 - c)
\]
Strategic Voter Model

\[ \dot{c} = (1 - 3c)T - \frac{r_0}{2} c(1 - c)(1 - 3c) \]

\[ \equiv -\frac{3r_0}{2} (c - c_-)(c - c_+)(c - c_3) \]

\[ c_3 = \frac{1}{3} \]

\[ c_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - 8x_0} \right) \]

\[ x_0 \equiv \frac{T}{r_0} \]

\[ \left[ \frac{c(t) - c_3}{c(0) - c_3} \right]^{\alpha_3} \left[ \frac{c(t) - c_+}{c(0) - c_+} \right]^{\alpha_+} \left[ \frac{c(t) - c_-}{c(0) - c_-} \right]^{\alpha_-} = e^{-3(c_+ - c_-)r_0 t/2} \]

\[ \alpha_\pm = \frac{1}{c_3 - c_\pm} \]

\[ \alpha_3 = \alpha_- - \alpha_+ = \frac{c_+ - c_-}{(c_3 - c_+)(c_3 - c_-)} \]
Strategic Voter Model

mean-field phase diagram

location of fixed point

Strategic Voter Model

mean-field phase diagram

location of fixed point
Simulations of Strategic Voter Model

(a) % vote over time

(b) Time in minority

(c) % vote over years
If $|x_2 - x_1| < 1$ compromise

If $|x_2 - x_1| > 1$ no interaction
The Opinion Distribution

\[ P(x,t) = \text{probability that agent has opinion } x \text{ at time } t \]

Fundamental parameter: \( \Delta \) the diversity (initial opinion range)

\( \Delta < 1: \) consensus
\( \Delta > 1: \) fragmentation

\[
\frac{\partial P(x, t)}{\partial t} = \int \int dx_1 dx_2 \ P(x_1, t)P(x_2, t) \times \left[ \delta \left( x - \frac{1}{2}(x_1 + x_2) \right) - \delta(x - x_1) \right]
\]

same as Maxwell model for inelastic collisions & inelastic collapse phenomena

\( w \sim e^{-\Delta t/2} \)

Ben-Naim and Krapivsky (2000)
Baldassarri, Marconi, Puglisi (2001)
Early time evolution (for $\Delta=4.2$)
integrate master equation rather than simulate!
Early time evolution (for $\Delta=4.2$)
integrate master equation rather than simulate!
Early time evolution (for $\Delta=4.2$)
integrate master equation rather than simulate!
Early time evolution (for $\Delta=4.2$)
integrate master equation rather than simulate!
Fragmentation Sequence

![Graph showing the relationship between final opinion and Δ (diversity). The graph indicates a significant decrease in final opinion with increasing diversity, labeled as 'minor'.]
Birth of Extremists

\[ t = 0 \]

\[ -(1 + \epsilon) \quad -1 \quad 1 \quad (1 + \epsilon) \]

separation:
\[ w = \epsilon = e^{-t_{\text{sep}}/2} \]

\[ w \approx e^{-t/2} \]
\[ m \approx \epsilon e^{-t} \]

\[ \rightarrow m(t_{\text{sep}}) \propto \epsilon^3 \]
Fragmentation Sequence

final opinion

\(\Delta (\text{diversity})\)

minor
major
Fragmentation Sequence

final opinion

Δ (diversity)

major
minor
central
## A Possible Realization

### 1993 Canadian Federal Election

<table>
<thead>
<tr>
<th>year</th>
<th>PQ</th>
<th>NDP</th>
<th>L</th>
<th>PC</th>
<th>SC</th>
<th>R/CA</th>
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<td>1979</td>
<td>26</td>
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<td>136</td>
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<td>40</td>
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<td>83</td>
<td>169</td>
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<tr>
<td>1993</td>
<td>54</td>
<td>9</td>
<td>177</td>
<td>2</td>
<td></td>
<td>52</td>
</tr>
</tbody>
</table>
The Dynamics of Persuasion

Sid Redner, Santa Fe Institute (physics.bu.edu/~redner)
CIRM, Luminy France, January 5-9, 2015

+ support from NSF

Modeling Consensus:
- introduction to the voter model
- voter model on complex networks
- voting with some confidence
- majority rule

Modeling Discord & Diversity:
- 3-state voter models
- strategic voting
- bounded compromise
- dynamics of social balance
- Axelrod model
Dynamics of Social Balance

friend

enemy

Investigate dynamical rules that promote evolution to a balanced state
Socially Balanced States

unfrustrated/balanced

frustrated/imbalanced

Social Balance

\{ a \text{ friend of my friend } \} \text{ is my friend; } \\
\{ a \text{ enemy of my enemy } \} \text{ is my enemy. }
Static properties of signed graphs:

Balanced states on complete graph must either be

- **utopia:** only friendly links
- **bipolar:** two mutually antagonistic cliques

Cartwright & Harary (1956)
Two Natural Evolution Rules

**Local Triad Dynamics:**
reduce imbalance in one triad by single update

\[
p: \text{amity parameter}
\]

→ Balance transition as a function of \( p \)

**Global Triad Dynamics:**
reduce global imbalance by single update

→ Outcome unknown

*Tantalizing connections to spin glasses & jamming phenomena*
Local Triad Dynamics on Arbitrary Networks
(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad

2. Reverse a single link so that the triad becomes balanced

   probability \( p \): unfriendly \( \rightarrow \) friendly;  probability \( 1-p \): friendly \( \rightarrow \) unfriendly

Fundamental parameter \( p \):

- \( p=1/3 \): flip a random link in the triad equiprobably
- \( p>1/3 \): predisposition toward tranquility
- \( p<1/3 \): predisposition toward hostility
The Evolving State

rate equation for the density of friendly links $\rho$:

\[
\frac{d\rho}{dt} = 3\rho^2(1 - \rho)[p - (1 - p)] + (1 - \rho)^3
\]

\[
= 3(2p - 1)\rho^2(1 - \rho) + (1 - \rho)^3
\]

\[
\rho(t) \sim \begin{cases} 
\rho_{\infty} + A e^{-Ct} & p < 1/2; \\
1 - \frac{1 - \rho_0}{\sqrt{1 + 2(1 - \rho_0)^2t}} & p = 1/2; \\
1 - e^{-3(2p - 1)t} & p > 1/2.
\end{cases}
\]

rapid approach to frustrated steady state
slow relaxation to utopia
rapid attainment of utopia
Simulations for a Finite Society

\[ p < \frac{1}{2}, \quad T_N \sim e^{N^2} \]

\[ p = \frac{1}{2}, \quad T_N \sim N^{4/3} \]

\[ p > \frac{1}{2}, \quad T_N \sim \frac{\ln N}{2p - 1} \]
Fate of a Finite Society

$p<1/2$: effective random walk picture

\[ D \propto N^2 \]

\[ u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim e^{vL_N/D} \sim e^{N^2} \]

\[ p>1/2: \text{ inversion of the rate equation} \]

\[ u = 1-\rho, \text{ the unfriendly link density} \]
\[ u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \quad \rightarrow \quad T_N \sim N^4 \]

Incorporating fluctuations as balance is approached:

\[ U = \frac{L}{\sqrt{t}} + \sqrt{L} t^{1/4} \]

Equating the 2 terms in U:

\[ T_N \sim L^{2/3} \sim N^{4/3} \]
Possible Application I: Long Beach Street Gangs

Nakamura, Tita, & Krackhardt (2007)

Gang relations

cool with
hate

Possible Application I: Long Beach Street Gangs

Gang relations

Nakamura, Tita, & Krackhardt (2007)

cool with
hate
Possible Application I: Long Beach Street Gangs

Nakamura, Tita, & Krackhardt (2007)

gang relations

violence frequency
Possible Application II: A Historical Lesson

3 Emperor’s League 1872-81

Triple Alliance 1882

German-Russian Lapse 1890

French-Russian Alliance 1891-94

Entente Cordiale 1904

British-Russian Alliance 1907
Axelrod Model

You:
- **car**: BMW, SUV, Ford, Trabant, bicycle
- **cuisine**: meat, vegetarian, vegan, fast weekly dieting
- **recreation**: skiing, hiking, fishing, hunting, running
- **politics**: leftist, rightist, anarchist, apathetic, fascist
- **abode**: suburb house, city apartment, homeless, pied-à-terre, city house

Me:
- **car**: BMW, SUV, Ford, Trabant, bicycle
- **cuisine**: meat, vegetarian, vegan, fast weekly dieting
- **recreation**: skiing, hiking, fishing, hunting, running
- **politics**: leftist, rightist, anarchist, apathetic, fascist
- **abode**: suburb house, city apartment, homeless, pied-à-terre, city house

Axelrod (1997)
Axelrod Model

You:
- **car**
  - BMW
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  - Ford
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  - bicycle
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  - vegetarian
  - vegan
  - fast weekly dieting
- **recreation**
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  - hiking
  - fishing
  - hunting
  - running
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  - rightist
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  - fascist
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F features

q traits

Axelrod (1997)
Axelrod Model

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  - city house

Axelrod (1997)
Axelrod's Simulation

$F=5$, $q=10$, 10x10 lattice

$t=0$  
$t=200$  
$t=400$  
$t=800$
Axelrod Model

simulations on 150 x 150 square lattice

small (F,q): consensus
large (F,q): fragmentation

fraction in largest cluster

F=10
1st-order transition

Castellano, Marsili & Vespignani (2000)
Axelrod Model

simulations on 150 x 150 square lattice

small \((F, q)\): consensus
large \((F, q)\): fragmentation

fraction in largest cluster

\(F=10\)
1st-order transition

\(F=2\)
2nd-order

\[ \frac{s_{\text{max}}}{L} \]

\(L = 50\)
\(L = 100\)
\(L = 150\)
Axelrod Model

simulations on 150 x 150 square lattice

fraction of active links

F=10

q=1

q=100

q=1000

q=10000

q=100000

t

0.0

0.2

0.4

0.6

0.8

1.0

1

10

100

1000

10000

100000

1000000

!
Axelrod Model with F=2 on 4-regular graph

\[ P_m \equiv \text{fraction of links with } m \text{ common features} \]

\[ m = 0, 2 \text{ inactive; } m = 1 \text{ active} \]

\[ q = q_c - \frac{1}{4} \]

\[ q = q_c - \frac{1}{4^k} \]

\[ P_0, P_1, P_2 \text{ bond densities} \]

\[ k = 1, 3, 5, 7 \]

\[ q = q_c + \frac{1}{4^k} \text{ long time scale} \]
Master Equations for Bond Densities

\[ \dot{P}_0 = \frac{z-1}{z} P_1 \left[ -\lambda P_0 + \frac{1}{2} P_1 \right] \]

\[ \dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \lambda P_0 - \frac{1}{2} (1 + \lambda) P_1 + P_2 \right] \]

\[ \dot{P}_2 = \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \frac{1}{2} \lambda P_1 - P_2 \right] \]

direct processes
direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \quad \rightarrow \quad \frac{\Delta P_1}{\Delta t} = -\frac{\frac{1}{2} P_1 / L}{1/N} = -\frac{P_1}{z}$$
Master Equations for Bond Densities

\[
\dot{P}_0 = \frac{z-1}{z} P_1 \left[ -\lambda P_0 + \frac{1}{2} P_1 \right]
\]

\[
\dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \lambda P_0 - \frac{1}{2} (1 + \lambda) P_1 + P_2 \right]
\]

\[
\dot{P}_2 = \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \frac{1}{2} \lambda P_1 - P_2 \right]
\]

indirect processes
direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \quad \rightarrow \quad \frac{\Delta P_1}{\Delta t} = -\frac{1}{2} \frac{P_1}{1/N} = -\frac{P_1}{z}$$

indirect processes for $\dot{P}_2$: $1 \rightarrow 2$

\[
\begin{array}{c c c c}
 a_1 & b_1 & a_1 & b_2 \\
 \hline
 1 & & 1 \\
 a_1 & b_1 \\
\end{array}
\]
direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \quad \rightarrow \quad \frac{\Delta P_1}{\Delta t} = -\frac{1}{2} \frac{P_1}{1/N} = -\frac{P_1}{z}$$

indirect processes for $\dot{P}_2$: 1 $\rightarrow$ 2

\[\text{Diagram:} \quad \text{a}_1 \quad \text{b}_1 \quad \text{a}_1 \quad \text{b}_2 \quad \text{a}_1 \quad \text{b}_1\]
direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \quad \Rightarrow \quad \frac{\Delta P_1}{\Delta t} = -\frac{1}{2} \frac{P_1}{1/N} = -\frac{P_1}{z}$$

indirect processes for $\dot{P}_2$: $1 \rightarrow 2$

```
\begin{align*}
a_1 & b_1 \\
\uparrow & \\
a_1 & b_2 \\
1 & \rightarrow 2 \\
\end{align*}
```
direct process for $\dot{P}_1$: choose random link

$$\Delta N_1 = -\frac{1}{2} P_1 \quad \Rightarrow \quad \frac{\Delta P_1}{\Delta t} = -\frac{1}{2} \frac{P_1/L}{1/N} = -\frac{P_1}{z}$$

indirect processes for $\dot{P}_2$: $1 \to 2$

rate $\frac{1}{2} P_1 \lambda \quad \Rightarrow \quad \frac{1}{2} P_1(q-1)^{-1}$
Master Equations for Bond Densities

\[
\dot{P}_0 = \frac{z-1}{z} P_1 \left[ -\lambda P_0 + \frac{1}{2} P_1 \right]
\]

\[
\dot{P}_1 = -\frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \lambda P_0 - \frac{1}{2} (1 + \lambda) P_1 + P_2 \right]
\]

\[
\dot{P}_2 = \frac{P_1}{z} + \frac{z-1}{z} P_1 \left[ \frac{1}{2} \lambda P_1 - P_2 \right]
\]

\[
d\tau \equiv \frac{z-1}{z} P_1 \, dt
\]

\[
x \equiv P_0
\]

\[
y \equiv P_1
\]

\[
P_2 = 1 - P_0 - P_1
\]
Master Equations for Bond Densities

\[ x' = -\lambda x + \frac{1}{2} y \]

\[ y' = \left(1 - \frac{1}{\eta}\right) + (\lambda - 1)x - \left(\frac{3+\lambda}{2}\right)y \]

\[ d\tau \equiv \frac{z-1}{z} P_1 \, dt \]

\[ x \equiv P_0 \]

\[ y \equiv P_1 \]

\[ P_2 = 1 - P_0 - P_1 \]
Master Equations for Bond Densities

\[ x' = -\lambda x + \frac{1}{2} y \]
\[ y' = (1 - \frac{1}{\eta}) + (\lambda - 1)x - \left(\frac{3+\lambda}{2}\right)y \]

\[ d\tau \equiv \frac{z-1}{z} P_1 \, dt \]
\[ x \equiv P_0 \]
\[ y \equiv P_1 \]
\[ P_2 = 1 - P_0 - P_1 \]
$x' = 0$

$x + y = 1$

$q_c - q$
\[ q = q_c - \frac{1}{4^k} \]

\[ k = -1, 1, 3, 5, 7 \]

\[ t_\infty \sim (q_c - q)^{-1/2} \]
\[ q = q_c - \frac{1}{4^k} \]
Axelrod Model with F=2

transition between steady state \((q<q_c)\) & fragmented static state \((q>q_c)\)

\quad q<q_c:\text{ very slow approach to steady state with time scale } \sim (q_c-q)^{-1/2}

long transient in which \(P_1 \sim (q_c-q)\) before steady state is reached
Some Closing Thoughts

Voter Model well characterized, but:
• consensus route incompletely understood on complex graphs
• generalizations, role of heterogeneity, role of internal beliefs,
• data-driven models

Models of Diversity & Discord
• bifurcation sequence in bounded compromise
• role of competing social interactions mostly unknown
• mathematical understanding of Axelrod model lacking
• data-driven models

Notes: physics.bu.edu/~redner:
   click the “slides from selected talks” link