Consensus and Deadlock in Opinion Dynamics

Sidney Redner, CNLS, LANL (on leave from Boston University)
collaborators: E. Ben-Naim, P. Chen, P. Krapivsky, V. Sood, F. Vazquez

Basic questions: What is the final state in prototypical opinion dynamics models with primarily ferromagnetic interactions?

How long does it take the reach the final state?

Models:

Voter model on heterogeneous graphs

Majority rule

Bounded compromise models

Spiteful extremists & accommodating centrists

Basic results:

Voter model: fast consensus on heterogeneous graphs

Majority rule: multiscale dynamics & slow consensus

Bounded compromise: rich political bifurcation sequence

Spiteful extremists: consensus versus deadlock

Voter Model

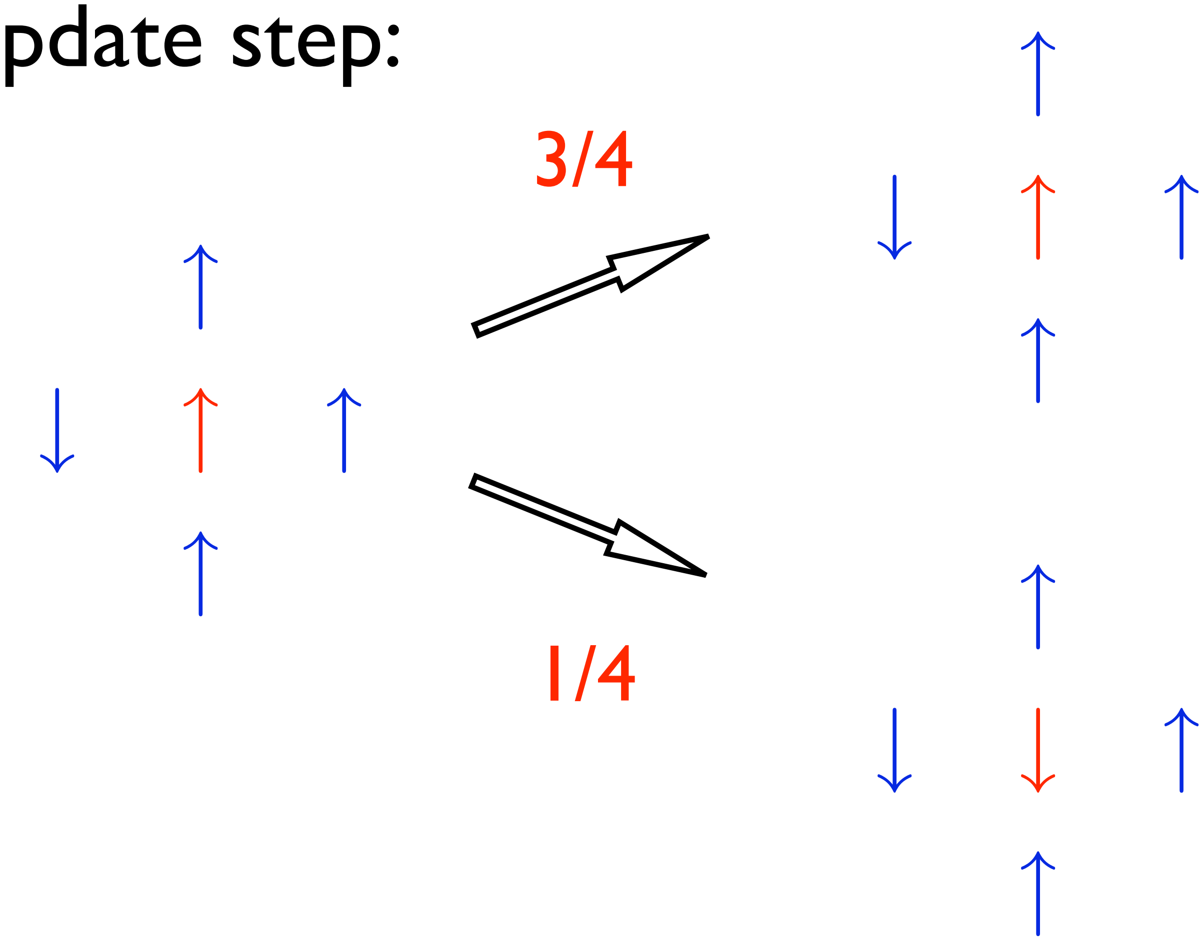
Liggett (1985)

0. Binary spin variable at each site
1. Pick a random spin
2. Assume state of randomly-selected neighbor

   each individual has zero self-confidence and adopts state of randomly-chosen neighbor

3. Repeat 1 & 2 until consensus *necessarily* occurs

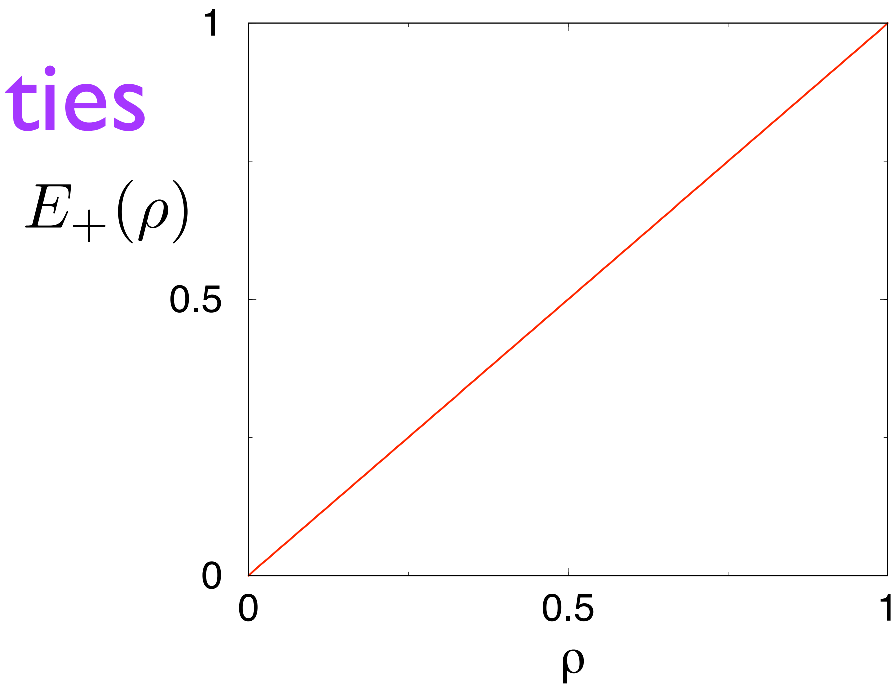
Example update step:



Voter model on regular lattices

1. Final state (exit) probabilities

follows from magnetization conservation



2. Dependence of consensus time on system size:

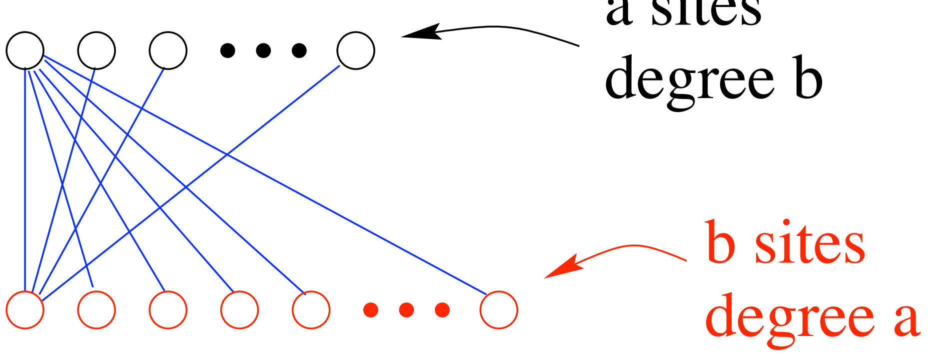
Liggett (1985), Krapivsky (1992)

dimension	consensus time
1	N^2
2	$N \ln N$
>2	N

Voter model on heterogeneous graphs

Catellano et al (2003), Suchecki et al (2004), Sood & SR (2005)

illustrative example: complete bipartite graph



pick site on the **a** sublattice ⇒ pick ↓ on **a** ⇒ pick ↑ on ***b*** sublattice

$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\frac{N_b}{b} - \frac{N_a}{a}\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\frac{N_a}{a} - \frac{N_b}{b}\frac{a-N_a}{a}\right]$$

Subgraph densities: $\rho_a = N_a/a,\ \rho_b = N_b/b \quad dt = 1/(a+b)$

$$\rho_{a,b}(t) = \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\, e^{-2t} + \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

$$\to \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

N.B.: magnetization is ***not*** conserved

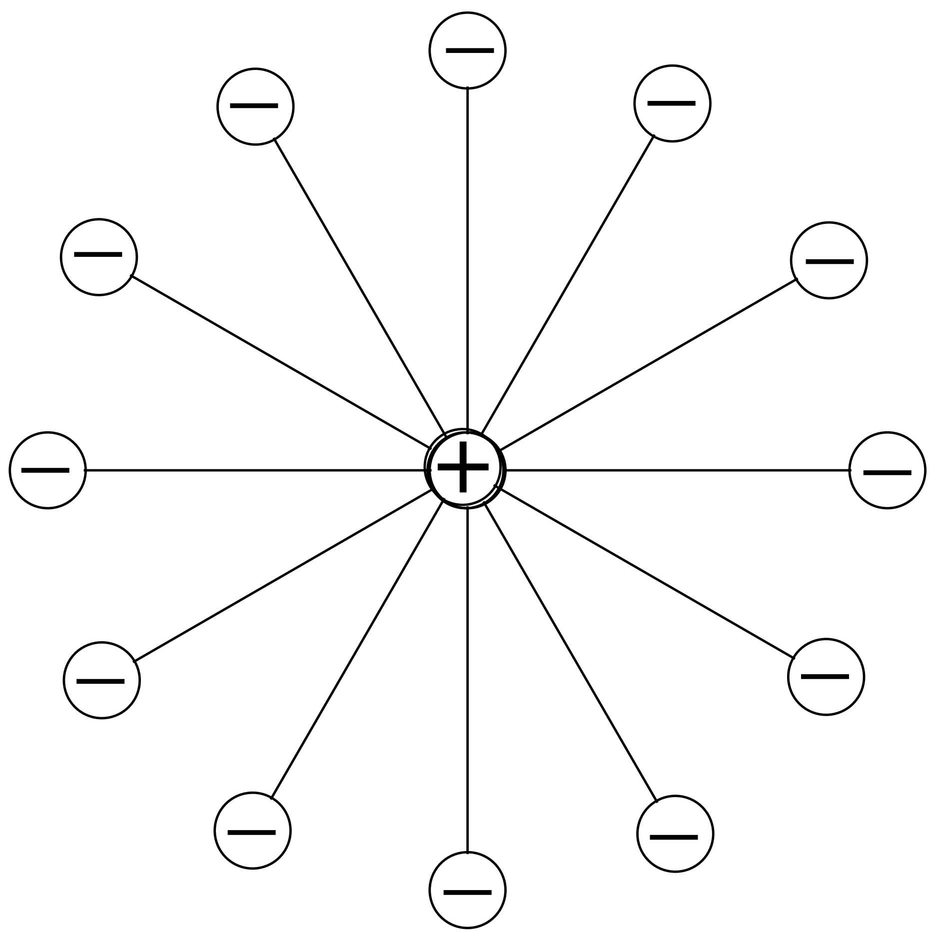
Exit probabilities:

$$E_+ = 1 - E_- = \frac{1}{2}[\rho_a(0) + \rho_b(0)].$$

Exit probabilities

$$E_+ = 1 - E_- = \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

extreme case: star graph



initial state: 1 plus, N minus

final state: all + with probability 1/2!

Mean consensus time

pick site on the a sublattice

pick ↓ on a

pick ↑ on *b* sublattice

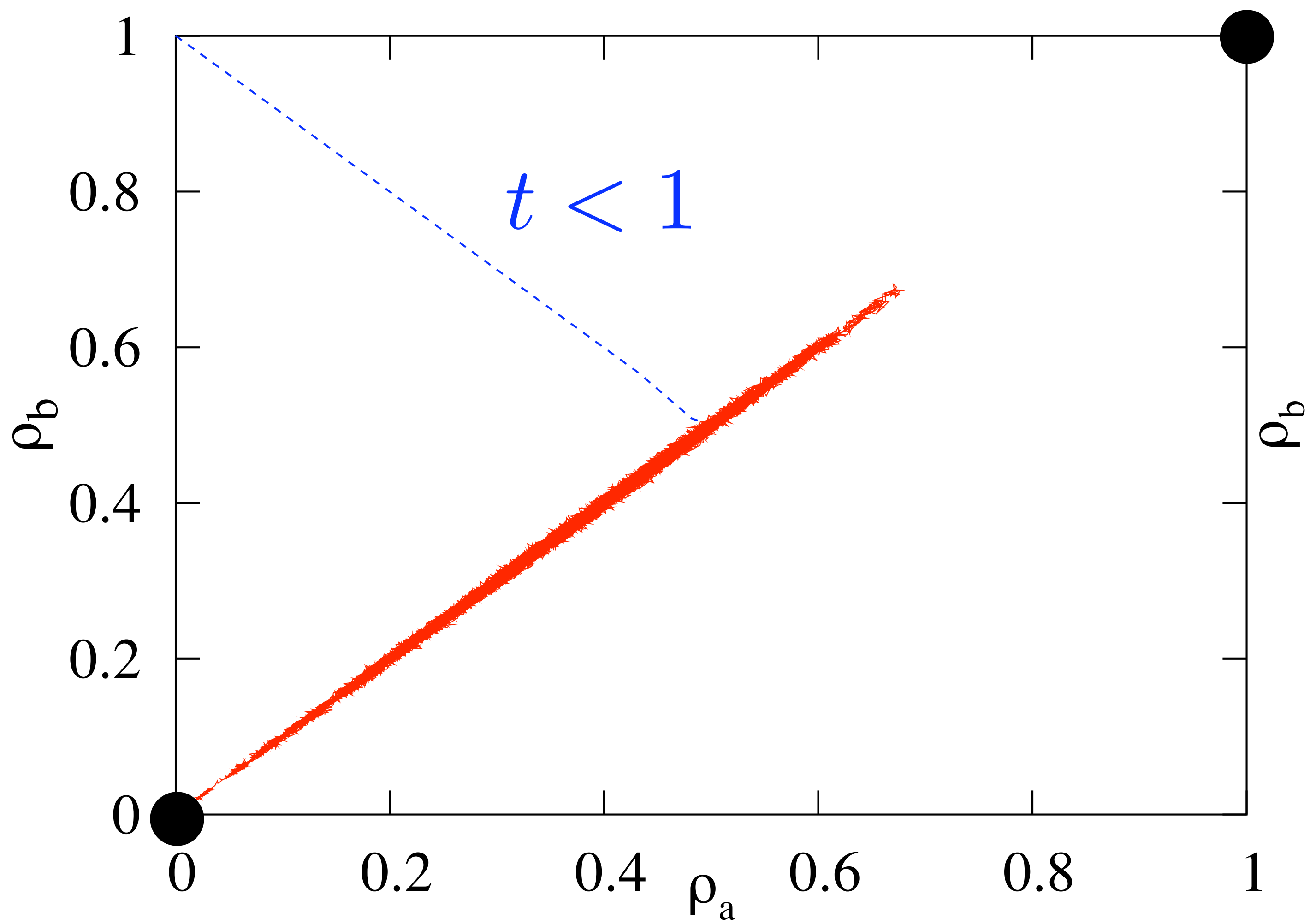
consensus time from new state

$$
\begin{aligned}
T(\rho_a,\rho_b) &= \frac{a}{a+b}(1-\rho_a)\rho_b\left[T(\rho_a+\frac{1}{a},\rho_b)+\delta t\right] \\
&+ \frac{a}{a+b}\rho_a(1-\rho_b)\left[T(\rho_a-\frac{1}{a},\rho_b)+\delta t\right] \\
&+ \frac{b}{a+b}(1-\rho_b)\rho_a\left[T(\rho_a,\rho_b+\frac{1}{b})+\delta t\right] \\
&+ \frac{b}{a+b}\rho_b(1-\rho_a)\left[T(\rho_a,\rho_b-\frac{1}{b})+\delta t\right] \\
&+ (1-\rho_a-\rho_b+2\rho_a\rho_b)\left[T(\rho_a,\rho_b)+\delta t\right],
\end{aligned}
$$

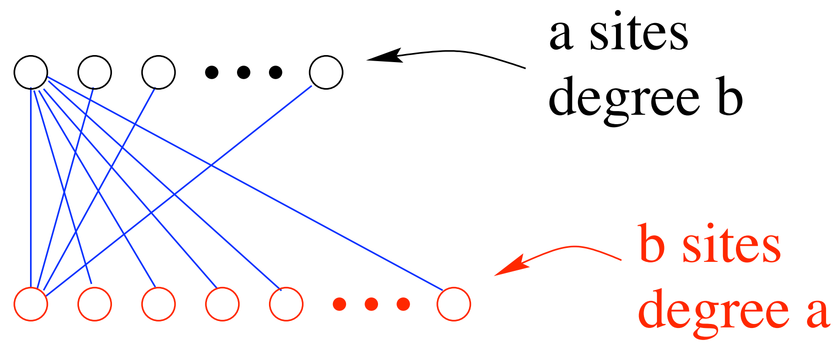
continuum limit:

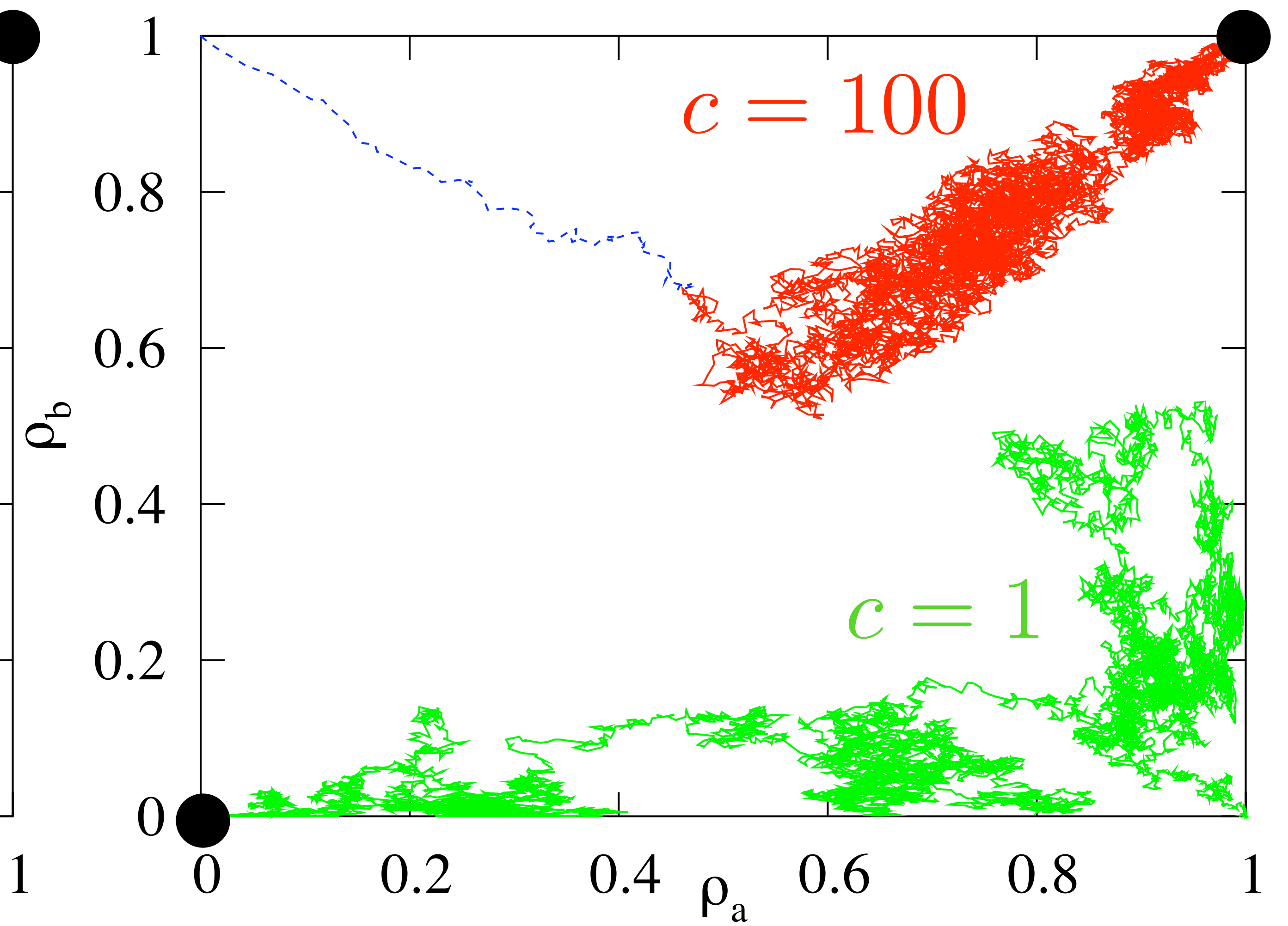
$$
\begin{aligned}
N\delta t &= (\rho_a-\rho_b)(\partial_a-\partial_b)T(\rho_a,\rho_b) \\
&-\frac{1}{2}(\rho_a+\rho_b-2\rho_a\rho_b)\left(\frac{1}{a}\partial_a^2+\frac{1}{b}\partial_b^2\right)T(\rho_a,\rho_b)
\end{aligned}
$$

Trajectories of single voter model realizations

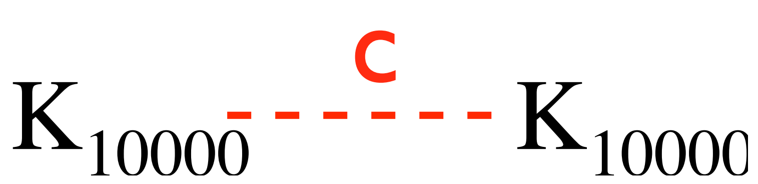


complete bipartite graph





two-clique graph



assuming $\rho_a = \rho_b$ and $\rho = (\rho_a + \rho_b)/2$:

equation of motion for T becomes:

$$\begin{aligned} N\delta t &= (\rho_a - \rho_b)(\partial_a - \partial_b)T(\rho_a, \rho_b) \\ &\quad -\frac{1}{2}(\rho_a + \rho_b - 2\rho_a\rho_b)\left(\frac{1}{a}\partial_a^2 + \frac{1}{b}\partial_b^2\right)T(\rho_a, \rho_b) \end{aligned}$$

$$\frac{1}{4}\rho(1-\rho)\left(\frac{1}{a}+\frac{1}{b}\right)\partial^2 T = -1$$

with solution:

$$T_{ab}(\rho) = -\frac{4ab}{a+b}\left[(1-\rho)\ln(1-\rho) + \rho\ln\rho)\right]$$

implication: $a = \mathcal{O}(1)$, $b = \mathcal{O}(N)$ (star graph), $T = \mathcal{O}(1)$

$a = \mathcal{O}(N)$, $b = \mathcal{O}(N)$ (symmetric graph), $T = \mathcal{O}(N)$

Arbitrary degree distribution network

$$n_j = \text{fraction of nodes with degree } j$$

$$\mu_m = \sum_j j^m n_j = m^{\text{th}} \text{ moment of degree distribution}$$

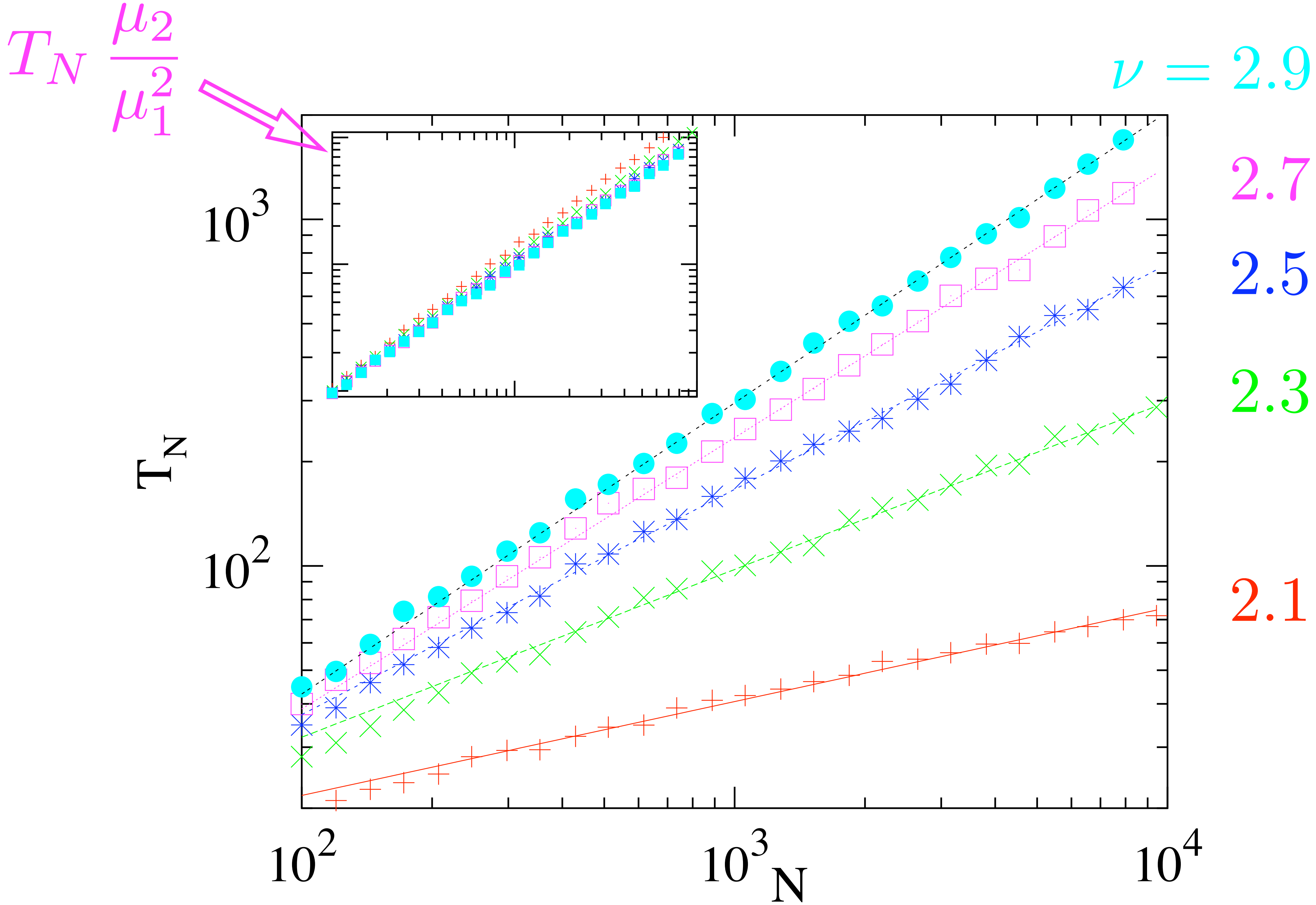
$$\omega = \frac{1}{\mu_1} \sum_j j n_j \rho_j = \text{degree-weighted up spin density}$$

Basic result: $$T_N(\omega) = -N \frac{\mu_1^2}{\mu_2} \left[(1-\omega)\ln(1-\omega) + \omega \ln \omega\right]$$

For power-law network: $(n_j \sim j^{-\nu})$

$$T_N \sim \begin{cases} N & \nu > 3, \\ N/\ln N & \nu = 3, \\ N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\ (\ln N)^2 & \nu = 2, \\ \mathcal{O}(1) & \nu < 2. \end{cases}$$

Consensus times for power-law degree distributions $n_j \sim j^{-\nu}$

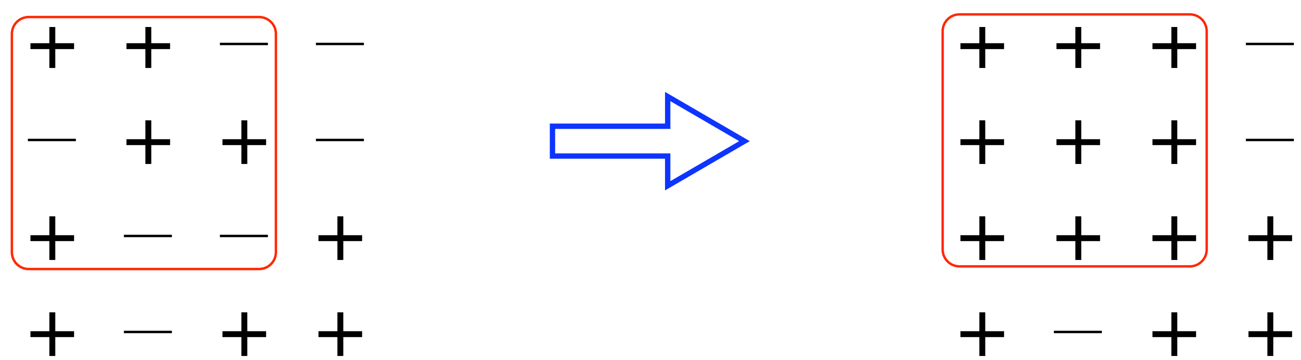


Basic results: quick consensus! universal scaling

Majority rule

Galam (1999), Krapivsky & SR (2003), Slanina & Lavicka (2003), Chen & SR (2005)

1. Pick a random group of G spins (with G odd).
2. All spins in G adopt the majority state.
3. Repeat until consensus necessarily occurs.



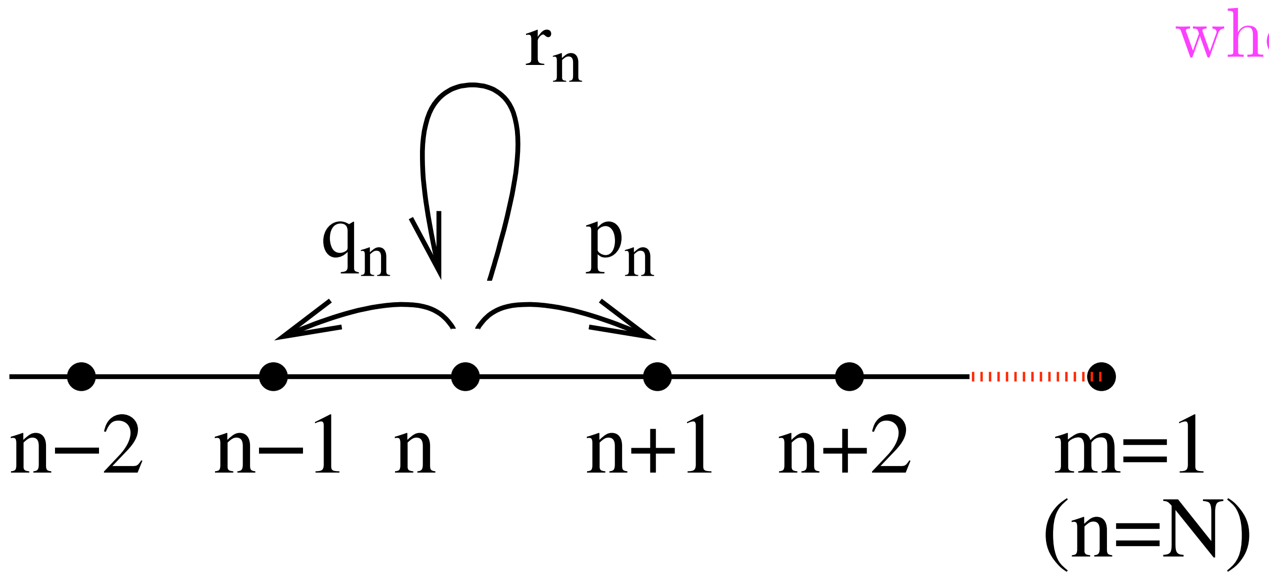
Basic questions: *1. Which final state is reached?*

2. What is the time until consensus?

Mean-field theory (for G=3)

$$E_n \equiv \text{exit probability to } m = 1 \text{ starting from } n \text{ plus spins}$$

$$= p_n E_{n+1} + q_n E_{n-1} + r_n E_n$$



where $p_n = \binom{3}{2}\binom{N-3}{n-2} \Big/ \binom{N}{n}$

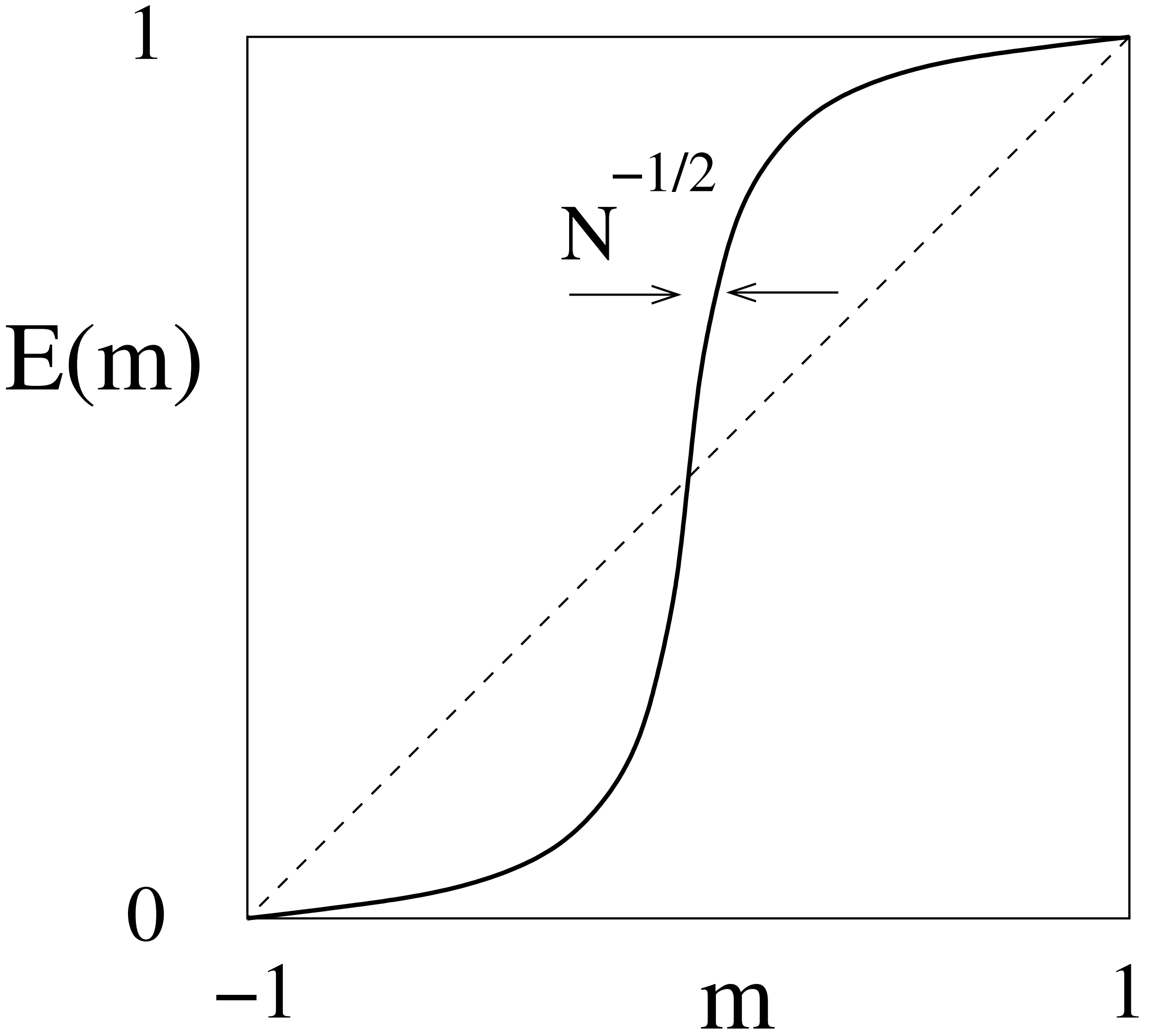
$q_n = \binom{3}{1}\binom{N-3}{n-1} \Big/ \binom{N}{n}$

$r_n = 1 - p_n - q_n$

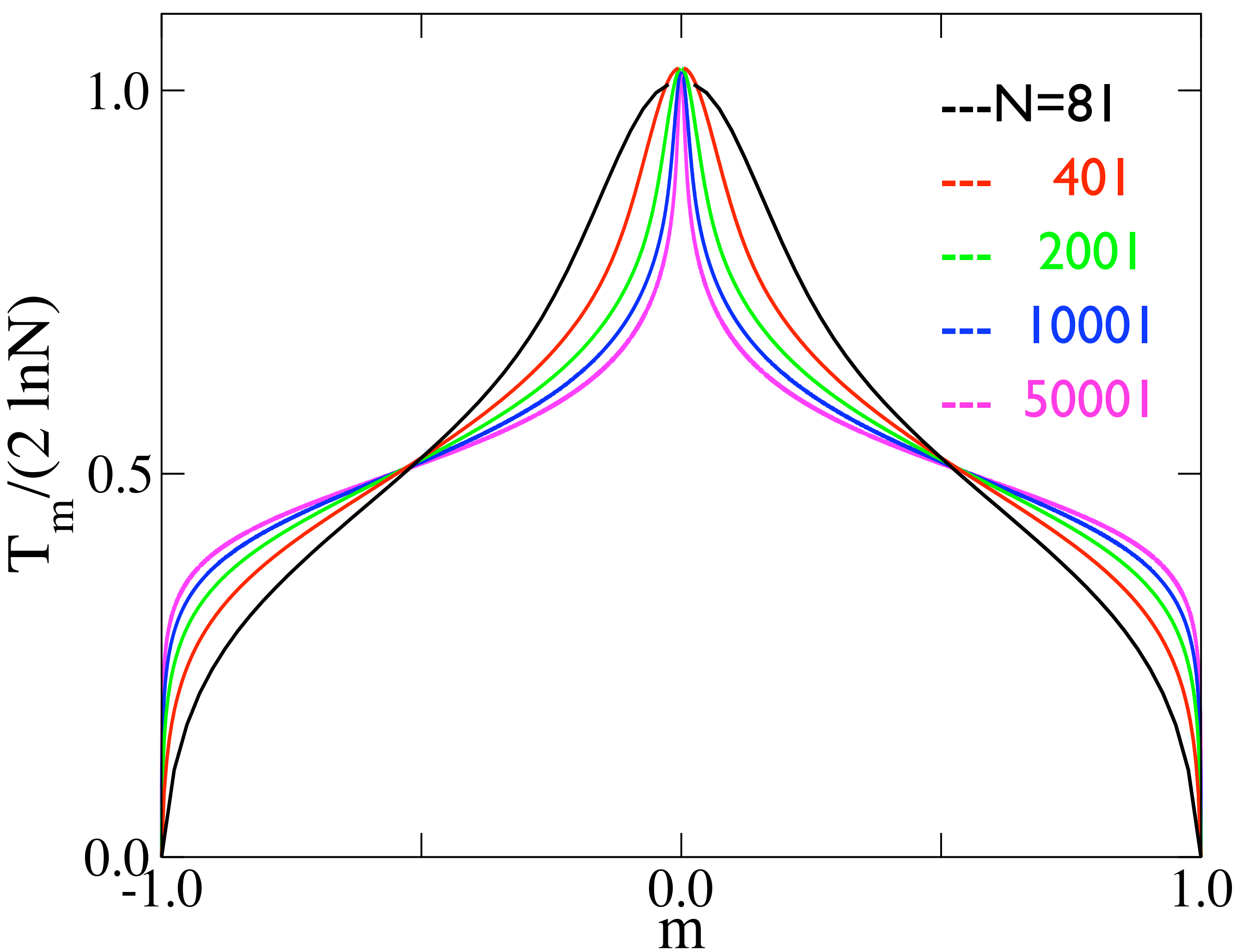
$$T_n \equiv \text{mean time to } m = 1 \text{ starting from } n \text{ plus spins}$$

$$= p_n(T_{n+1} + \delta t) + q_n(T_{n-1} + \delta t) + r_n(T_n + \delta t)$$

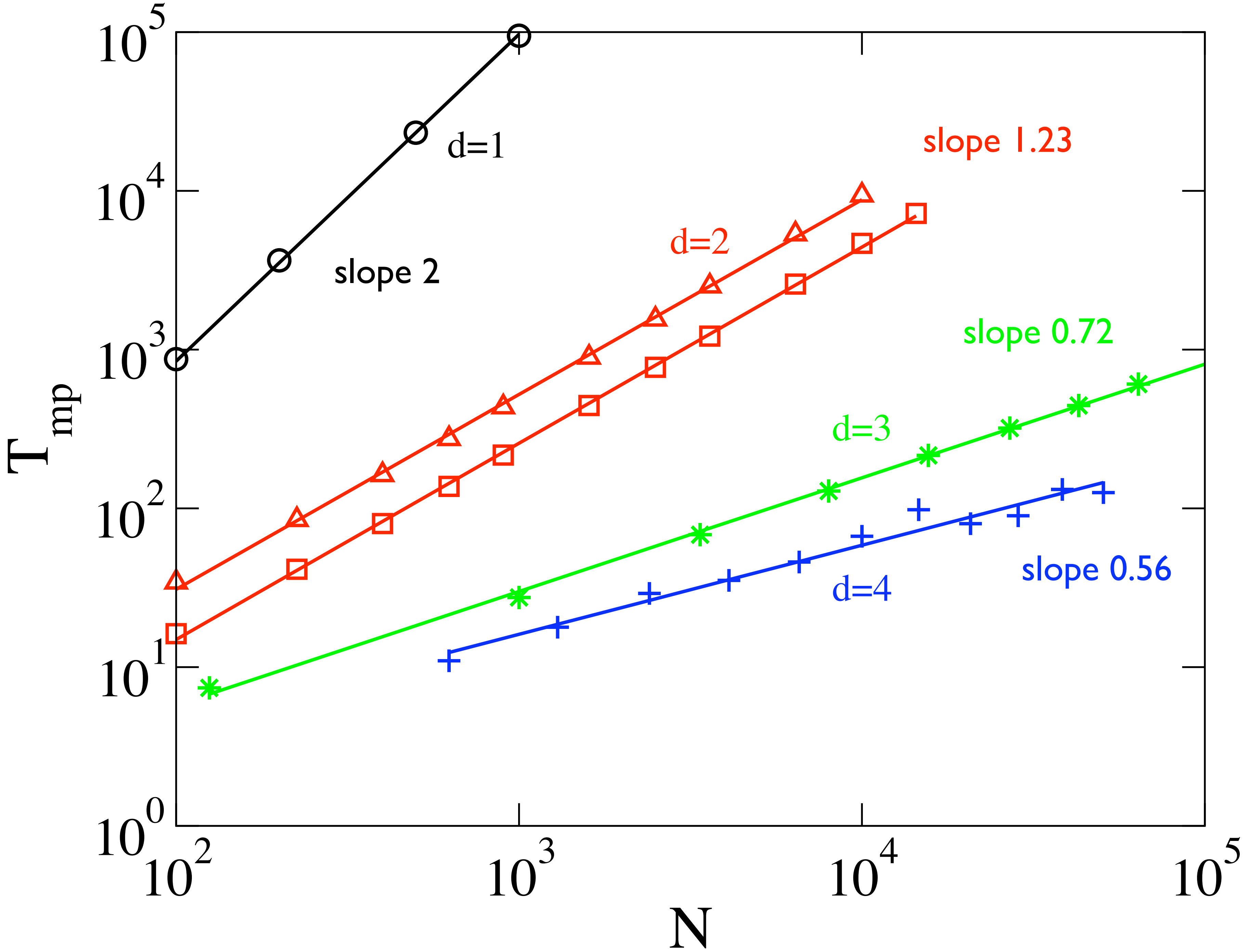
Exit probability
(schematic)



Consensus time
(data)



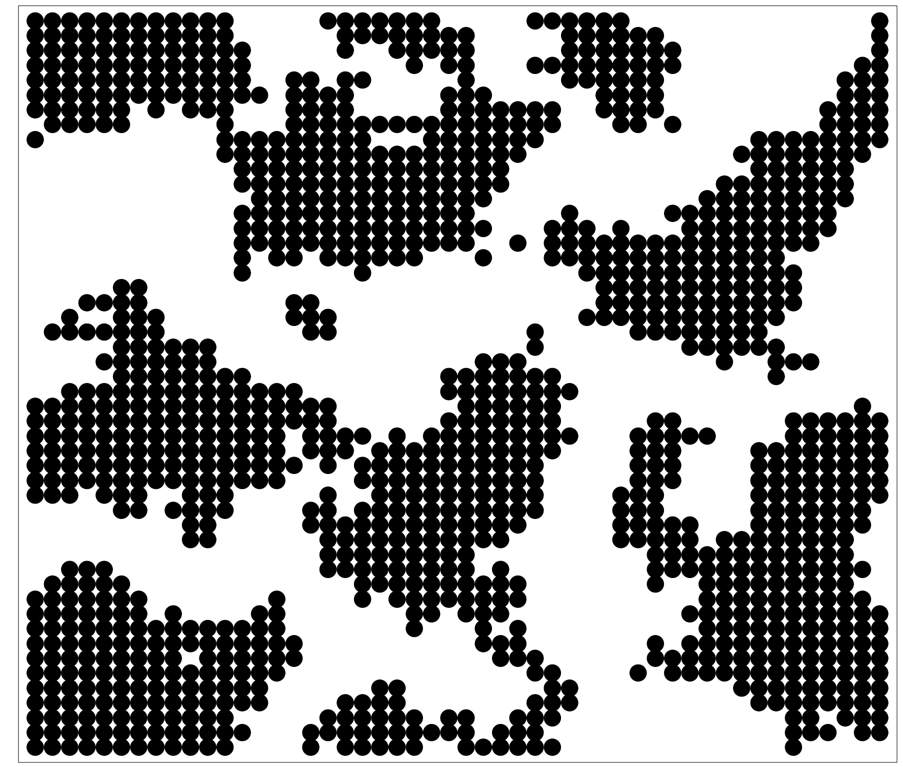
Consensus time for finite spatial dimensions



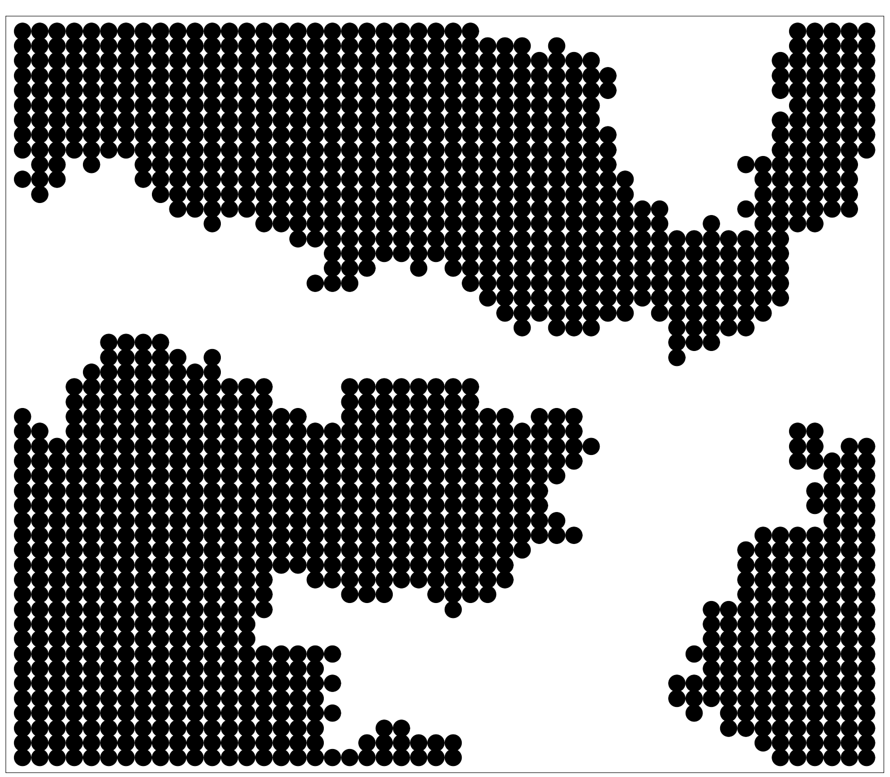
Critical dimension appears to be >4!

Anomalous dynamics in 2d: stripes ~33% of the time!

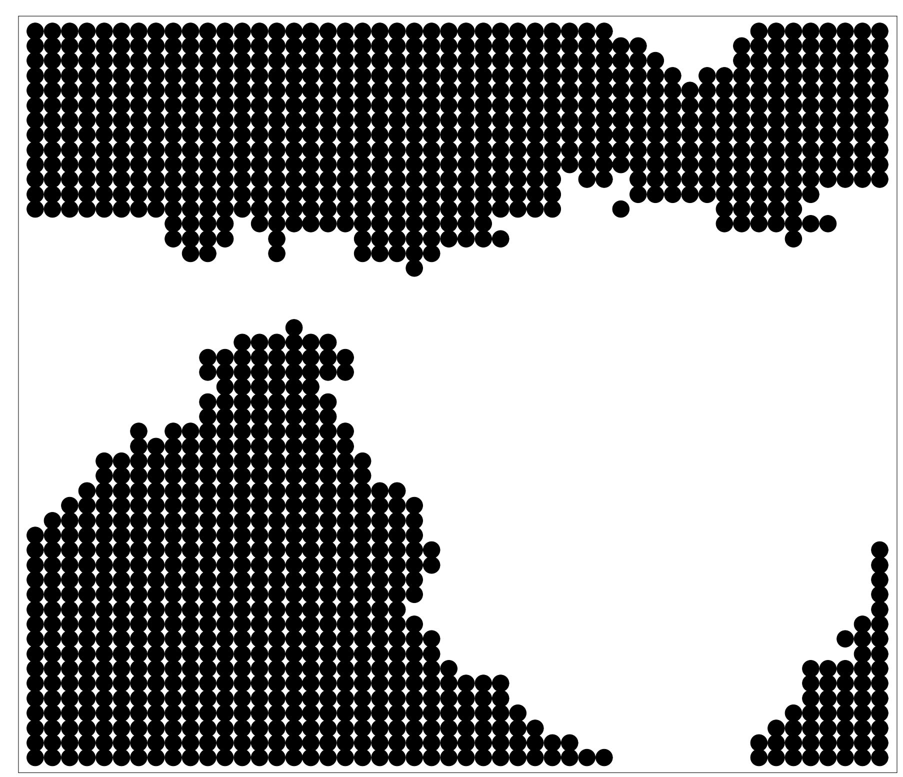
t=0



t=5

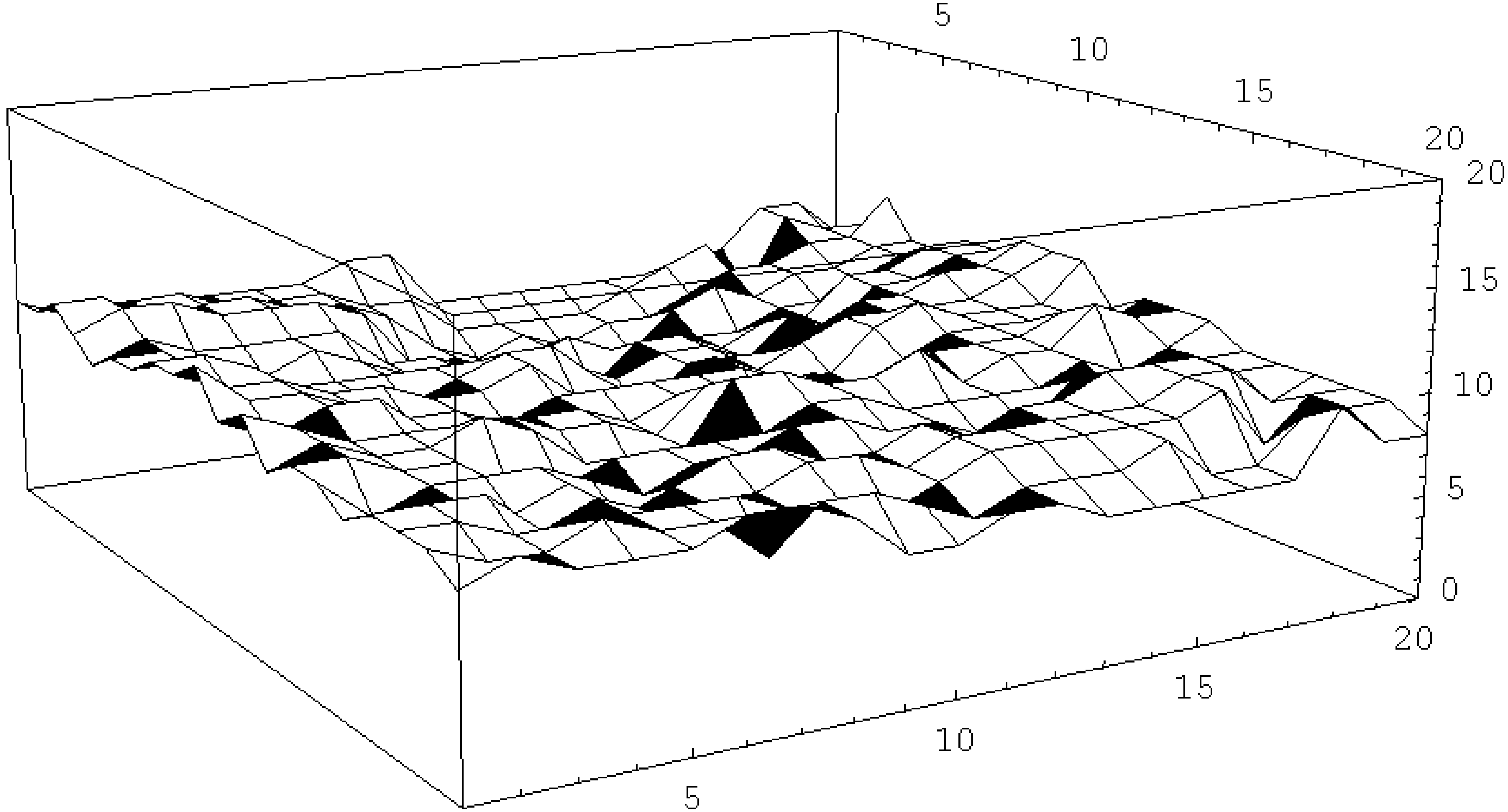


t=20

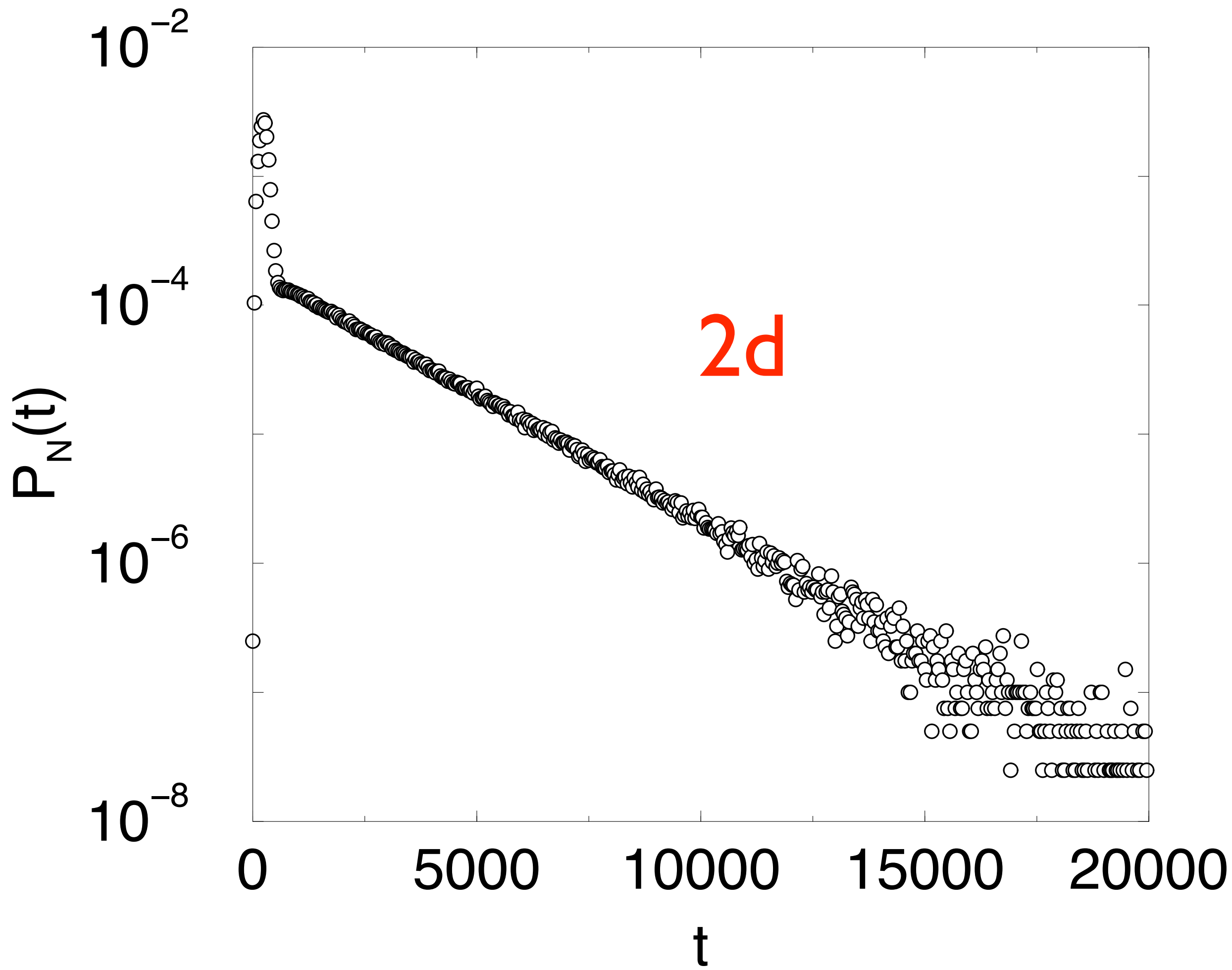


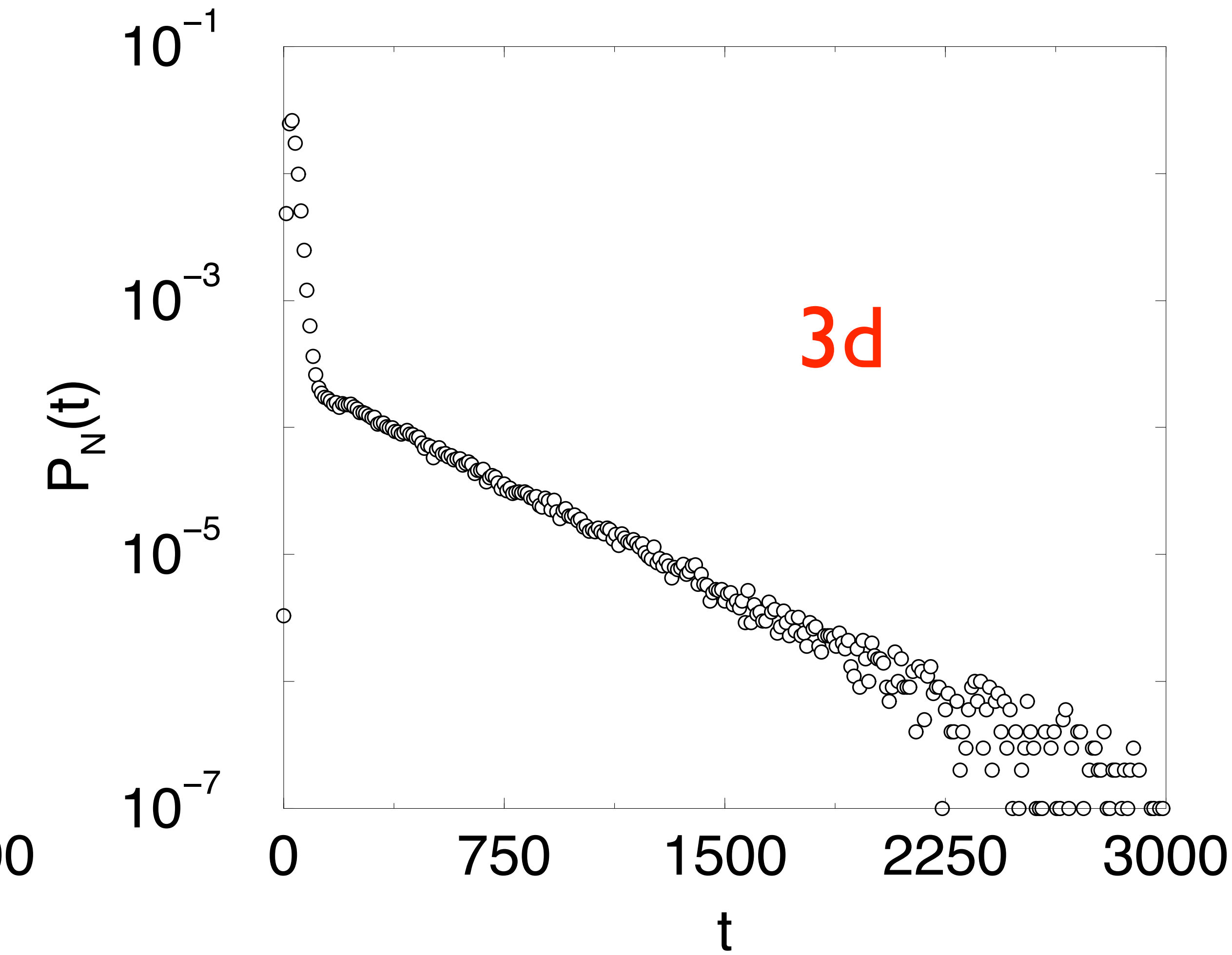
t=80

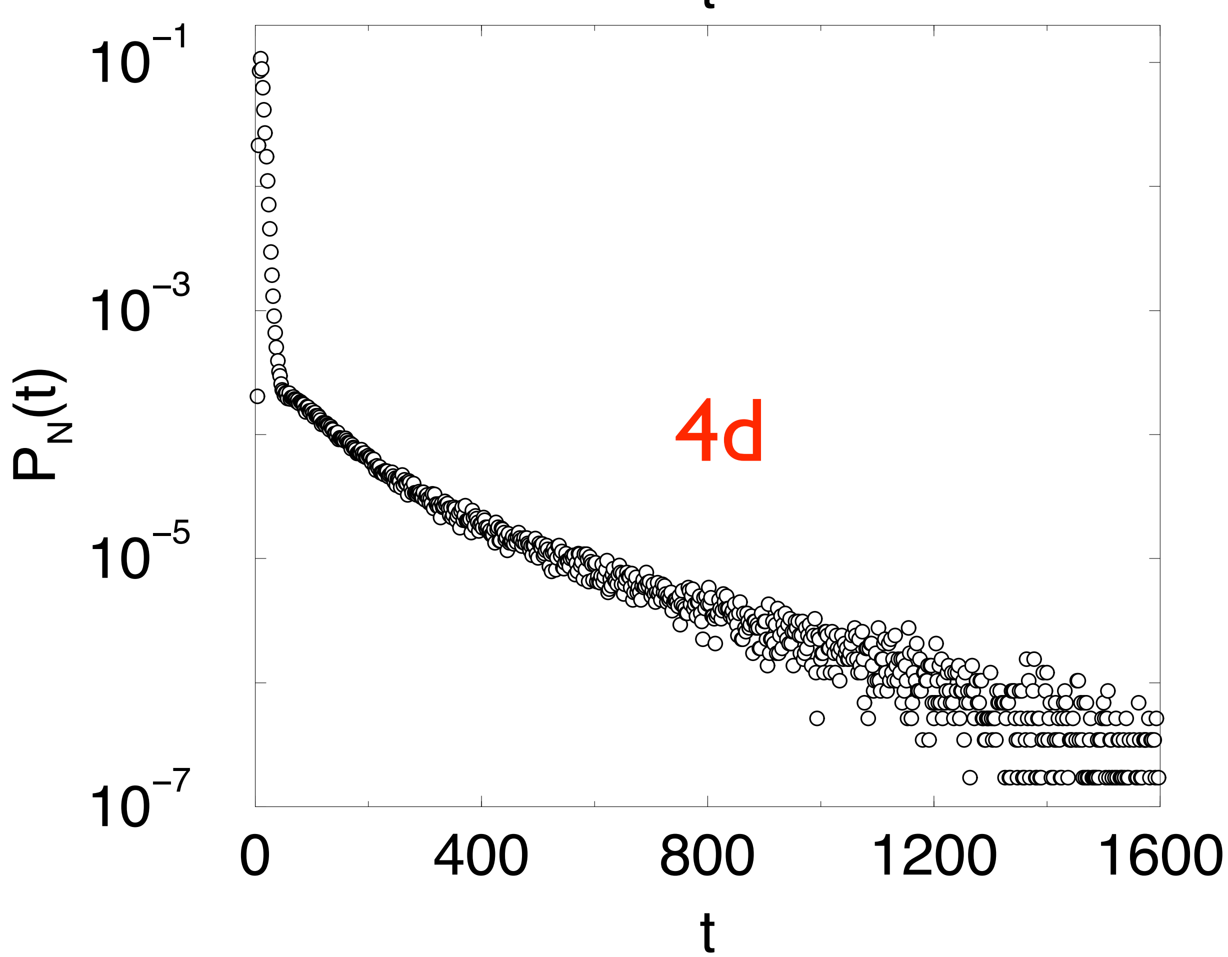
Slab formation in 3d ~8% of the time



Consensus time distribution



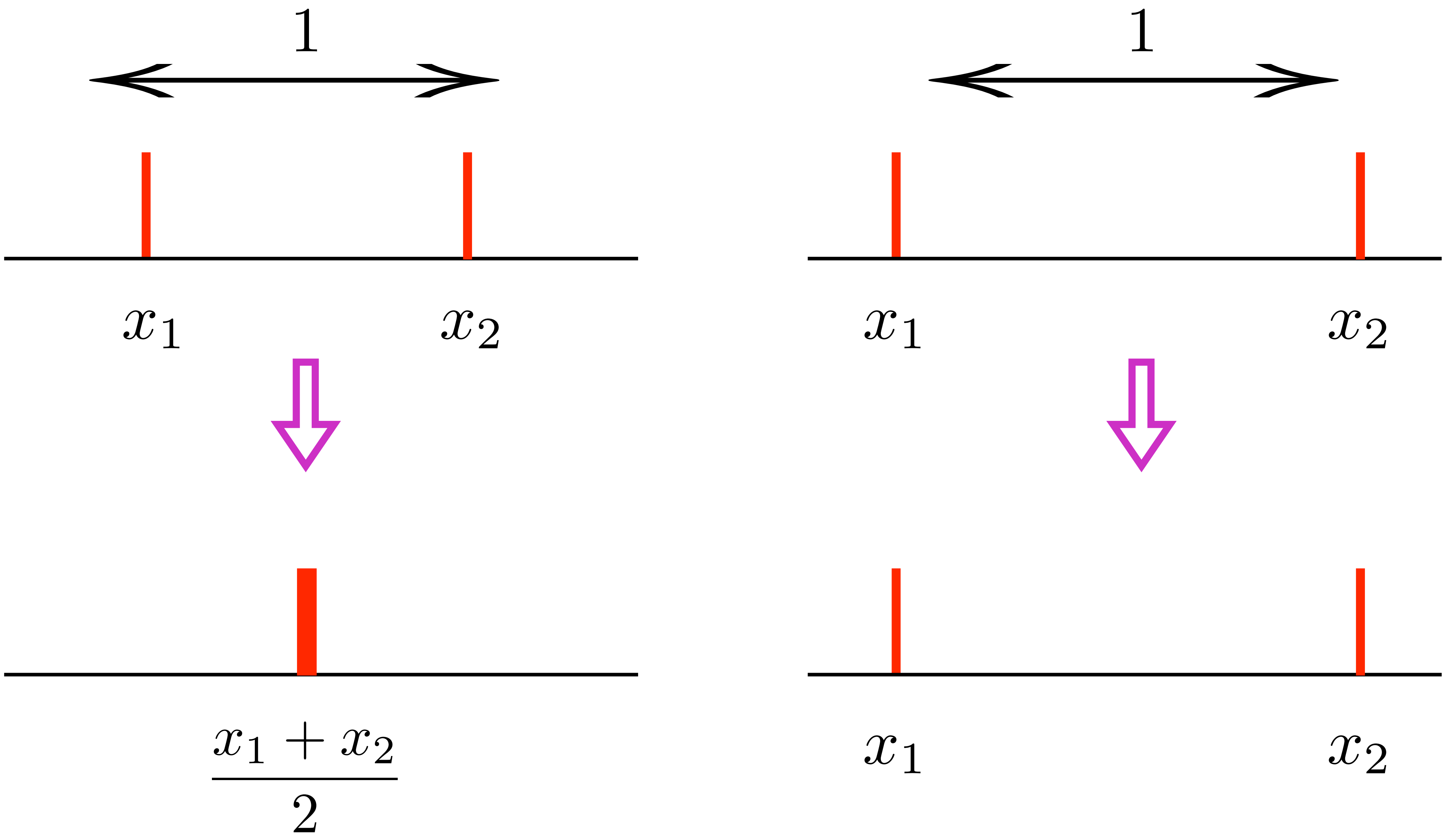




multiscale relaxation
to final consensus

Bounded compromise model

Deffuant et al (2000)



If $|x_2 - x_1| < 1$ compromise

If $|x_2 - x_1| > 1$ no interaction

Master equation

Fundamental parameter: Δ, the initial opinion range

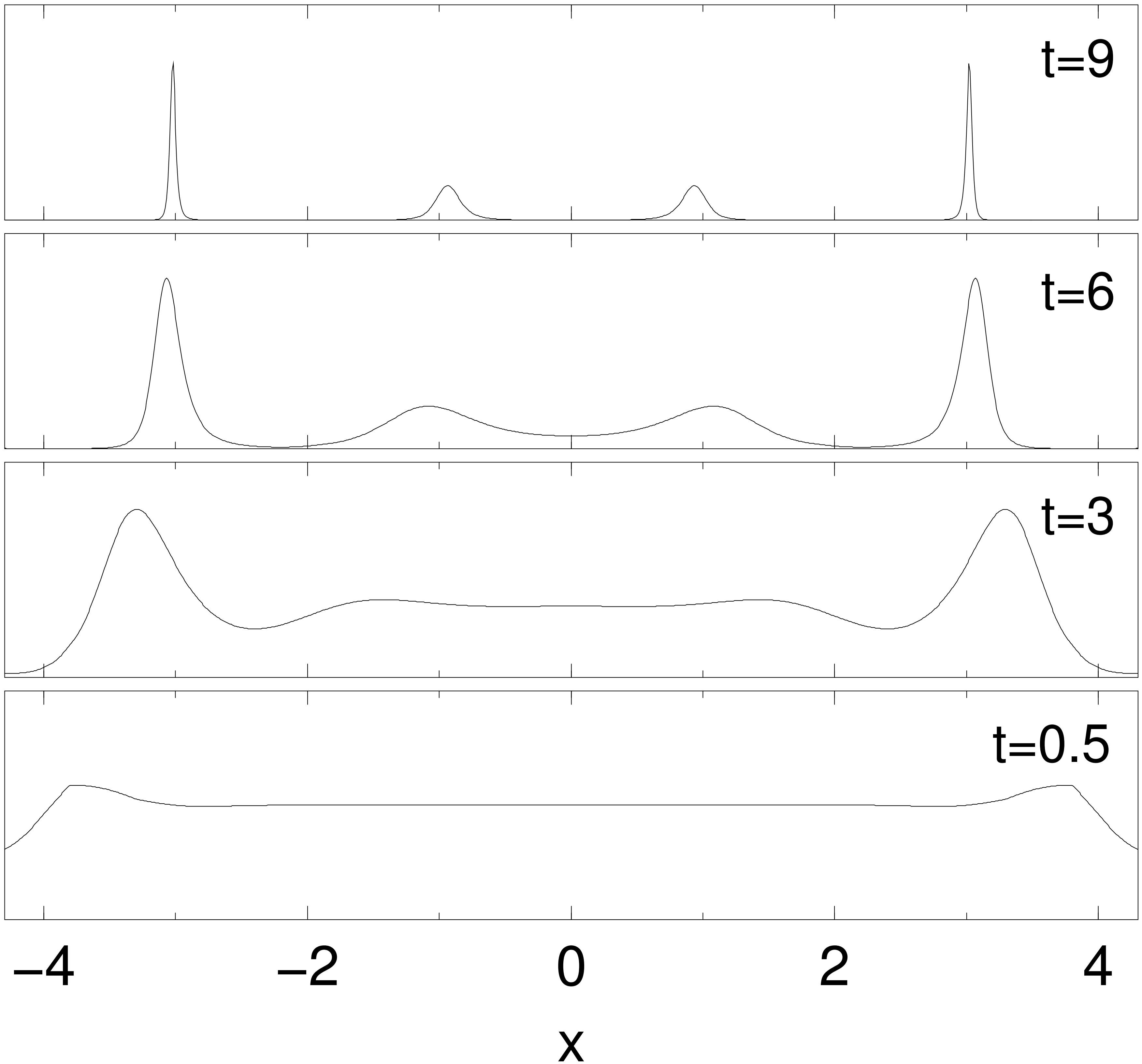
Basic observable: P(x,t) = probability that agent has opinion x

$$\frac{\partial P(x,t)}{\partial t} = \int\int_{|x_1-x_2|<1} dx_1\, dx_2\, P(x_1,t)P(x_2,t) \times \left[\delta\left(x - \frac{x_1+x_2}{2}\right) - \delta(x-x_1)\right]$$

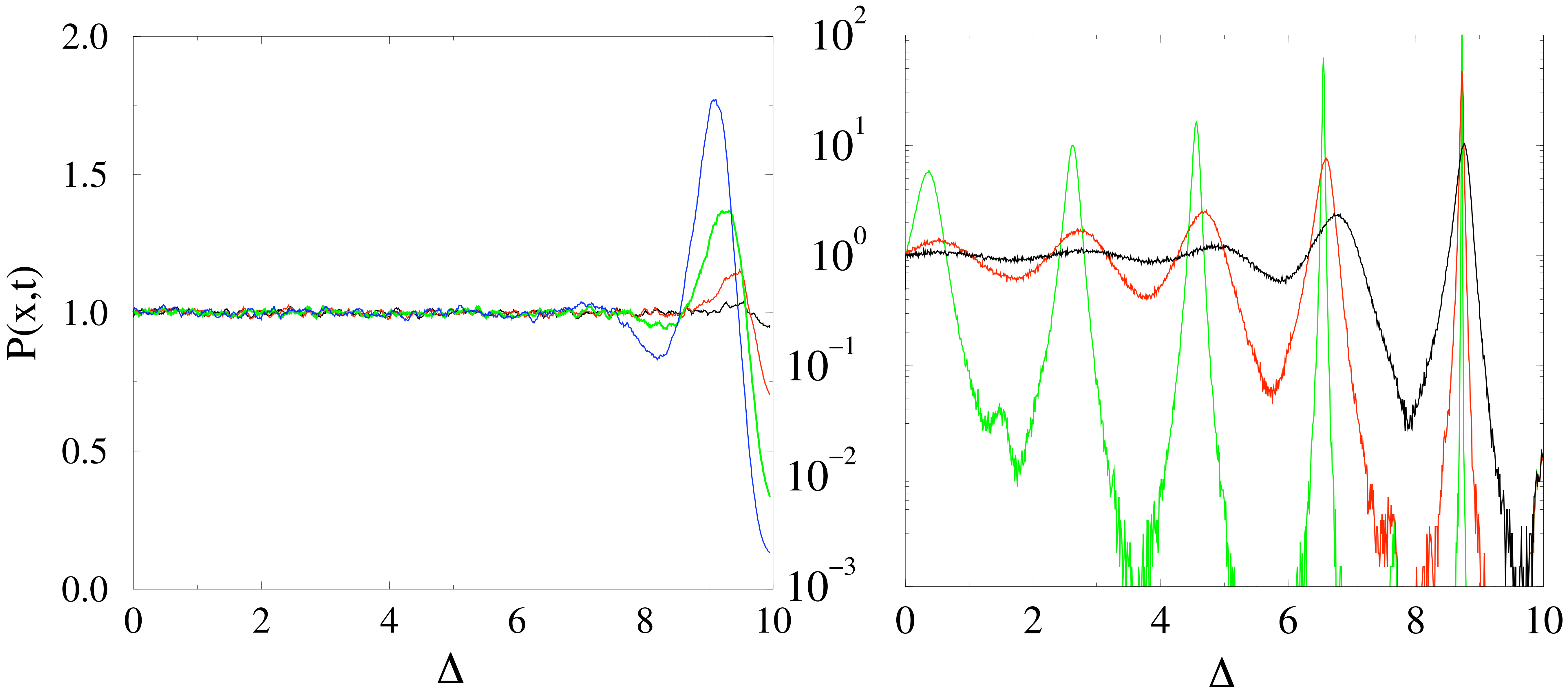
Δ<1: eventual consensus

Δ>1: disjoint "parties"

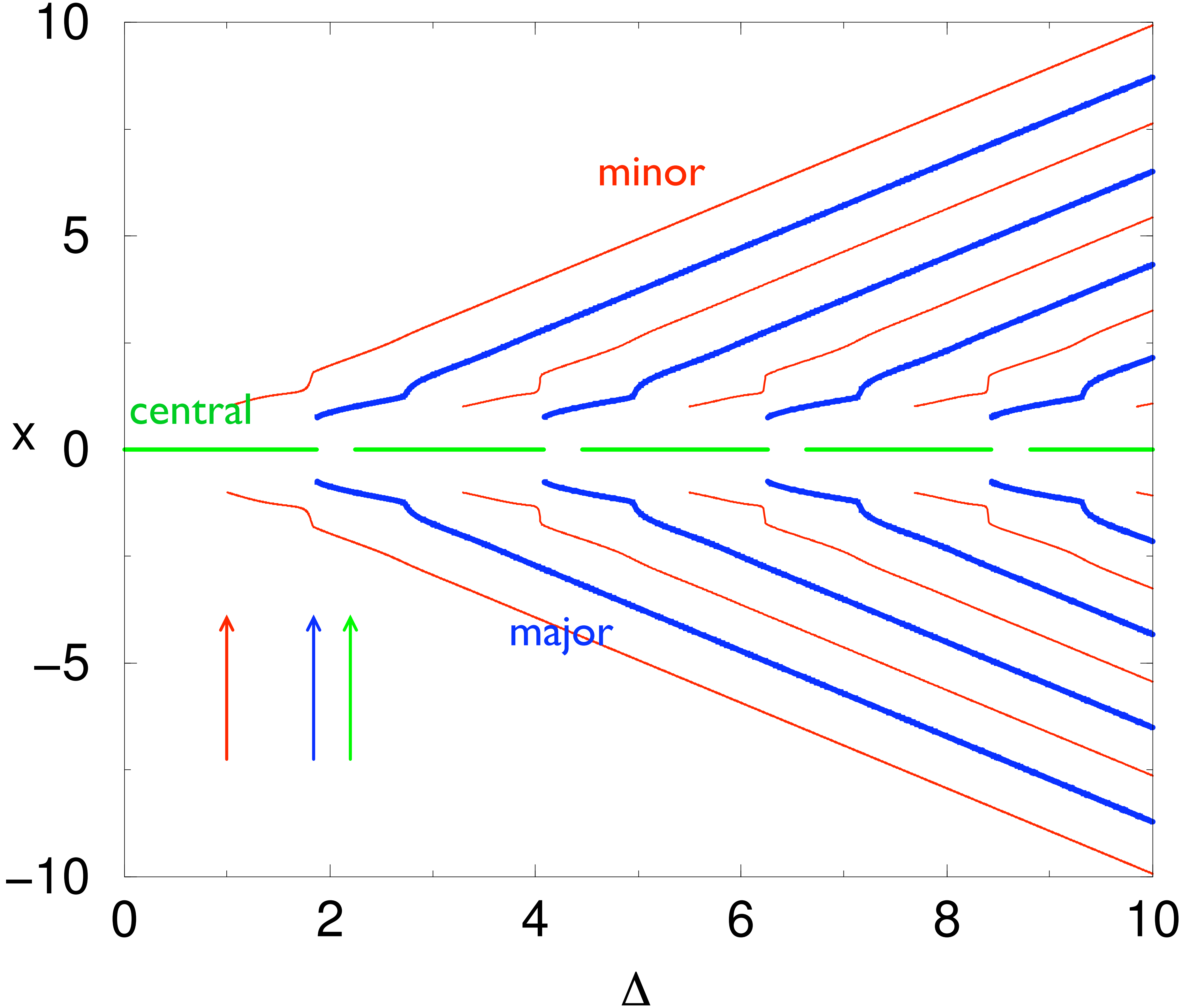
Early time evolution (for Δ=4)



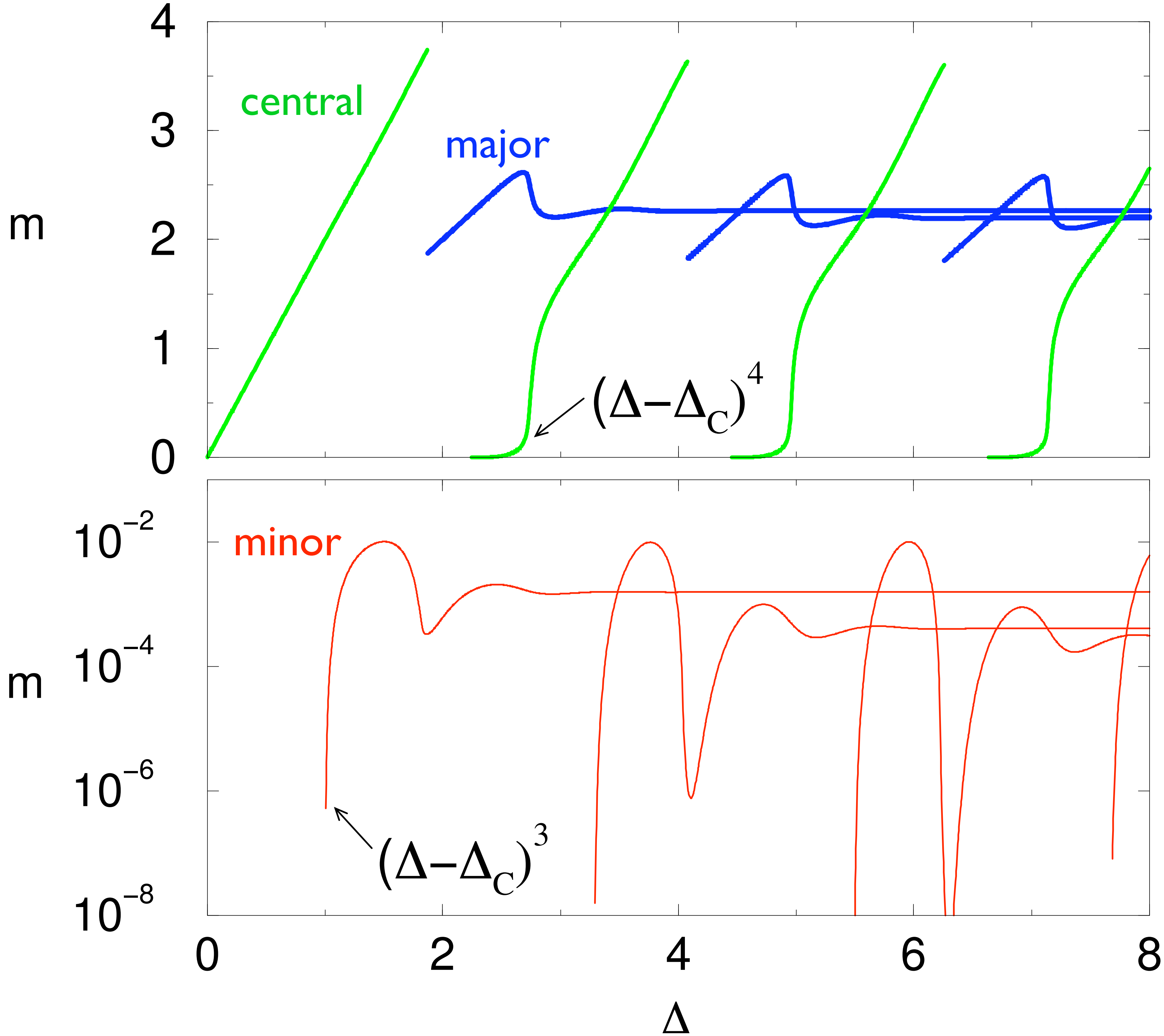
Early time evolution (for Δ=10)



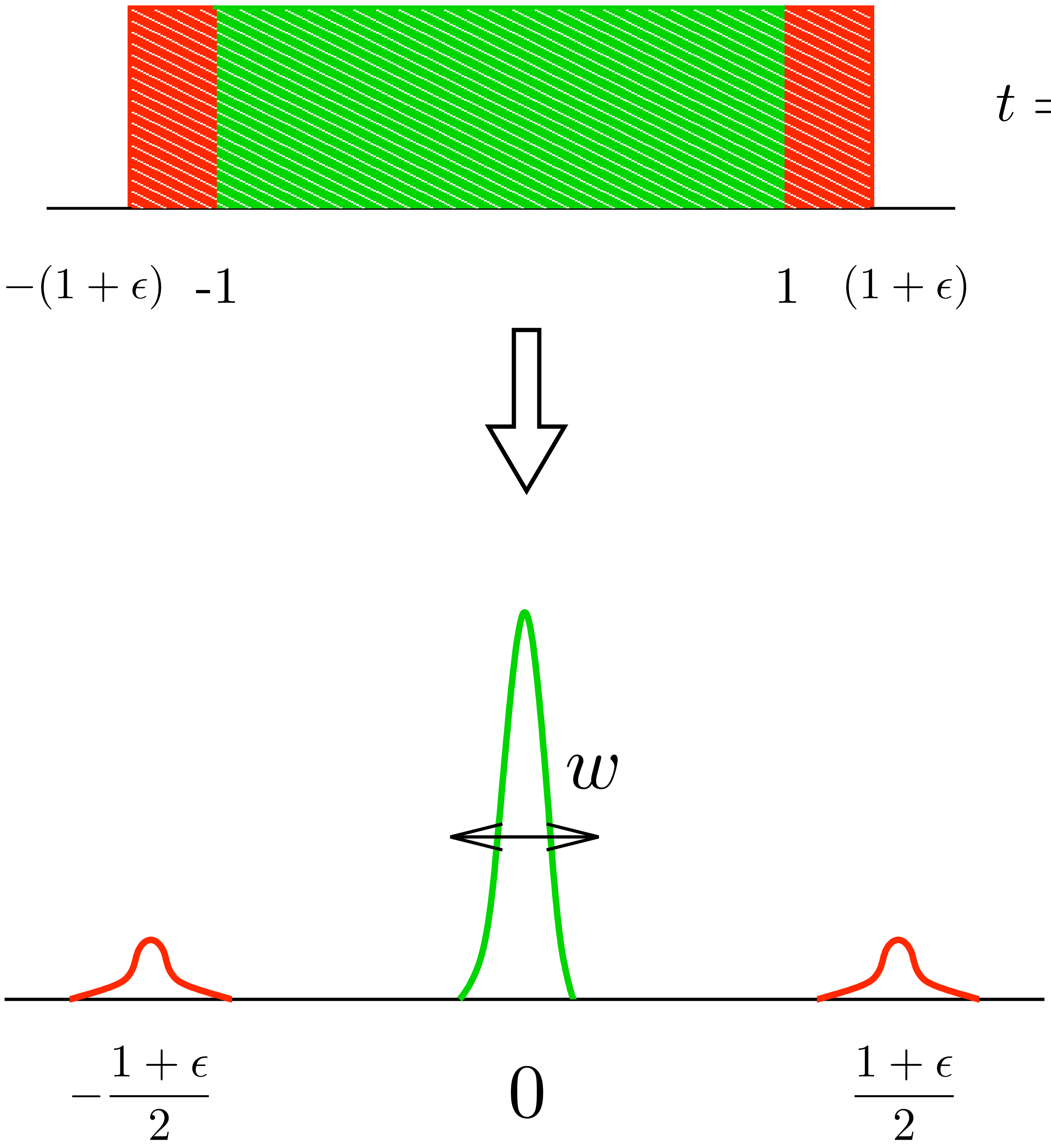
Bifurcation sequence



Cluster masses versus Δ



Minor cluster bifurcations



major cluster: $w \approx e^{-t/2}$

minor cluster: $\dot{m} = -m$

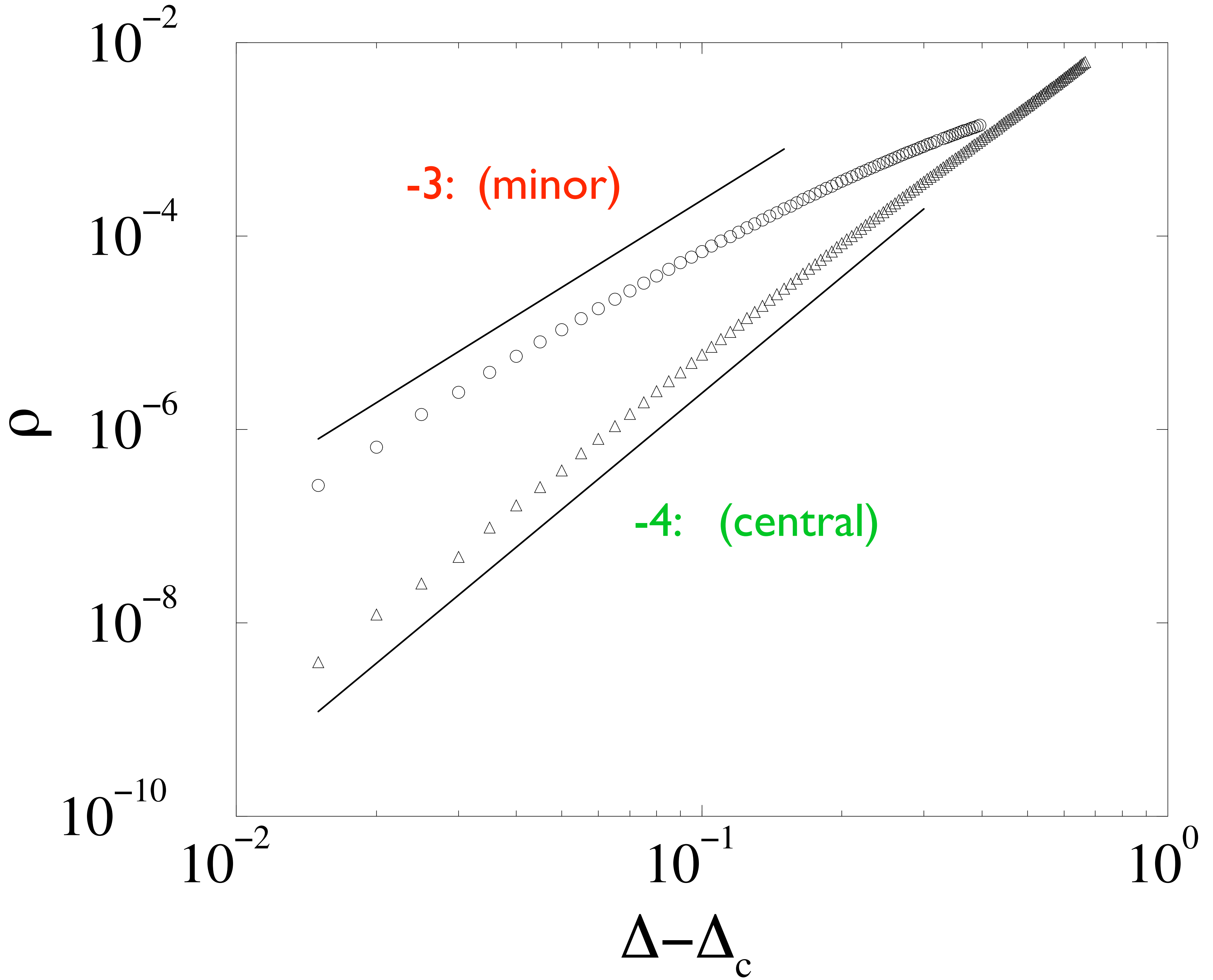
$\rightarrow m(t) = m(0)\, e^{-t} = \epsilon\, e^{-t}$

separation:

$w = \epsilon = e^{-t_{\text{sep}}/2}$

$\rightarrow m(t_{\text{sep}}) \propto \epsilon^3$

Cluster masses near bifurcations



Spiteful extremist model

Vazquez & SR (2004) -- inspired by the bounded confidence model of Deffuant et al (2000)

0. 3-state variable at each site: $- \quad 0 \quad +$
1. Pick a random spin
2. Assume state of neighbor if compatible
3. Repeat until either consensus or frozen final state

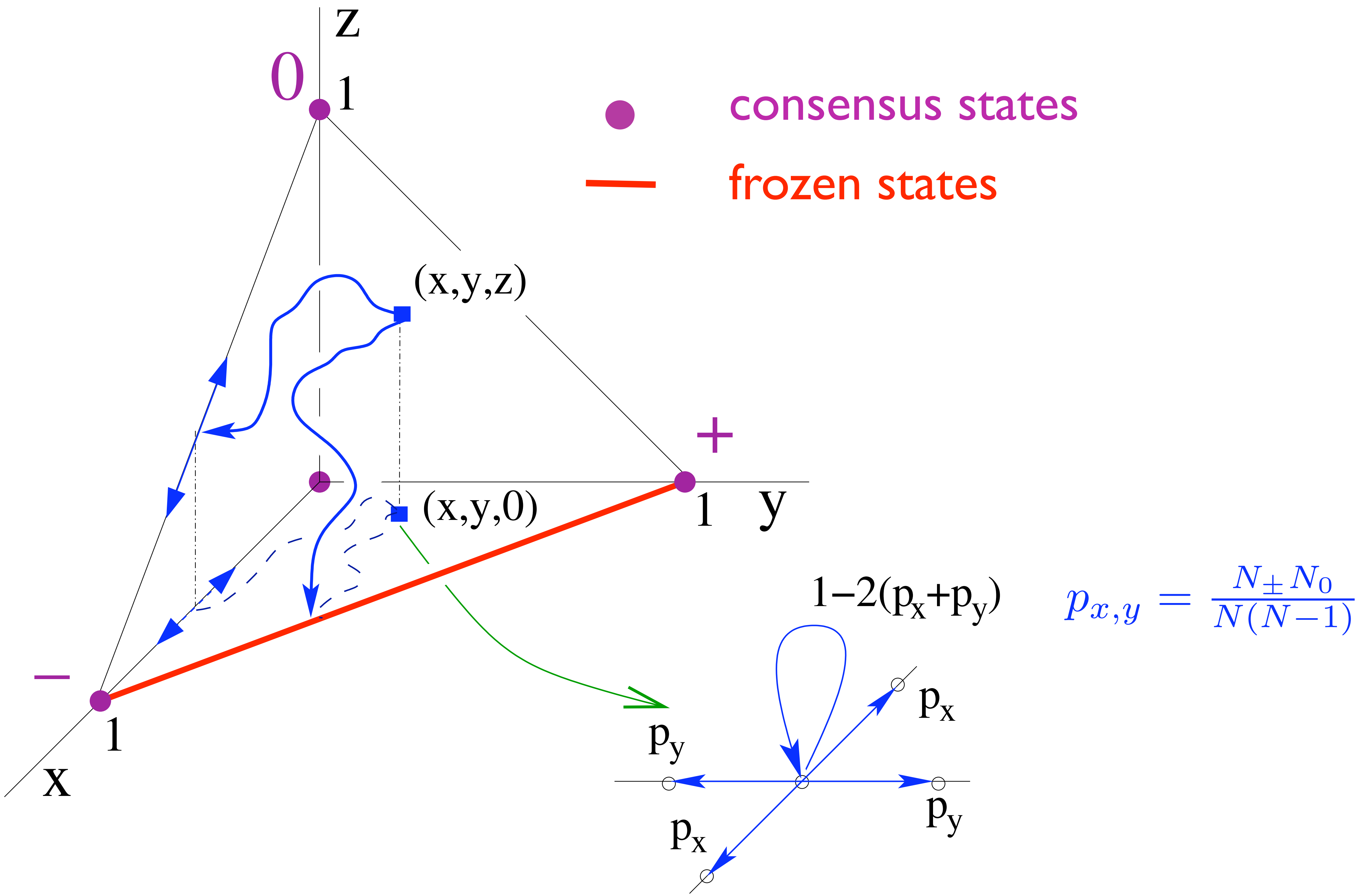
$$+0 \to ++, \quad +0 \to 00 \qquad -0 \to --, \quad -0 \to 00$$

compatible

$$+- \to +-$$

incompatible

Evolution in composition space



Probability to reach frozen final state

$F(x,y) = \text{probability to reach frozen state from } (x,y)$

− +

recursion formula:

$$
\begin{aligned}
F(x,y) &= p_x[F(x-\delta,y)+F(x+\delta,y)] \\
&+ p_y[F(x,y-\delta)+F(x,y+\delta)] \\
&+ [1-2(p_x+p_y)]F(x,y)
\end{aligned}
$$

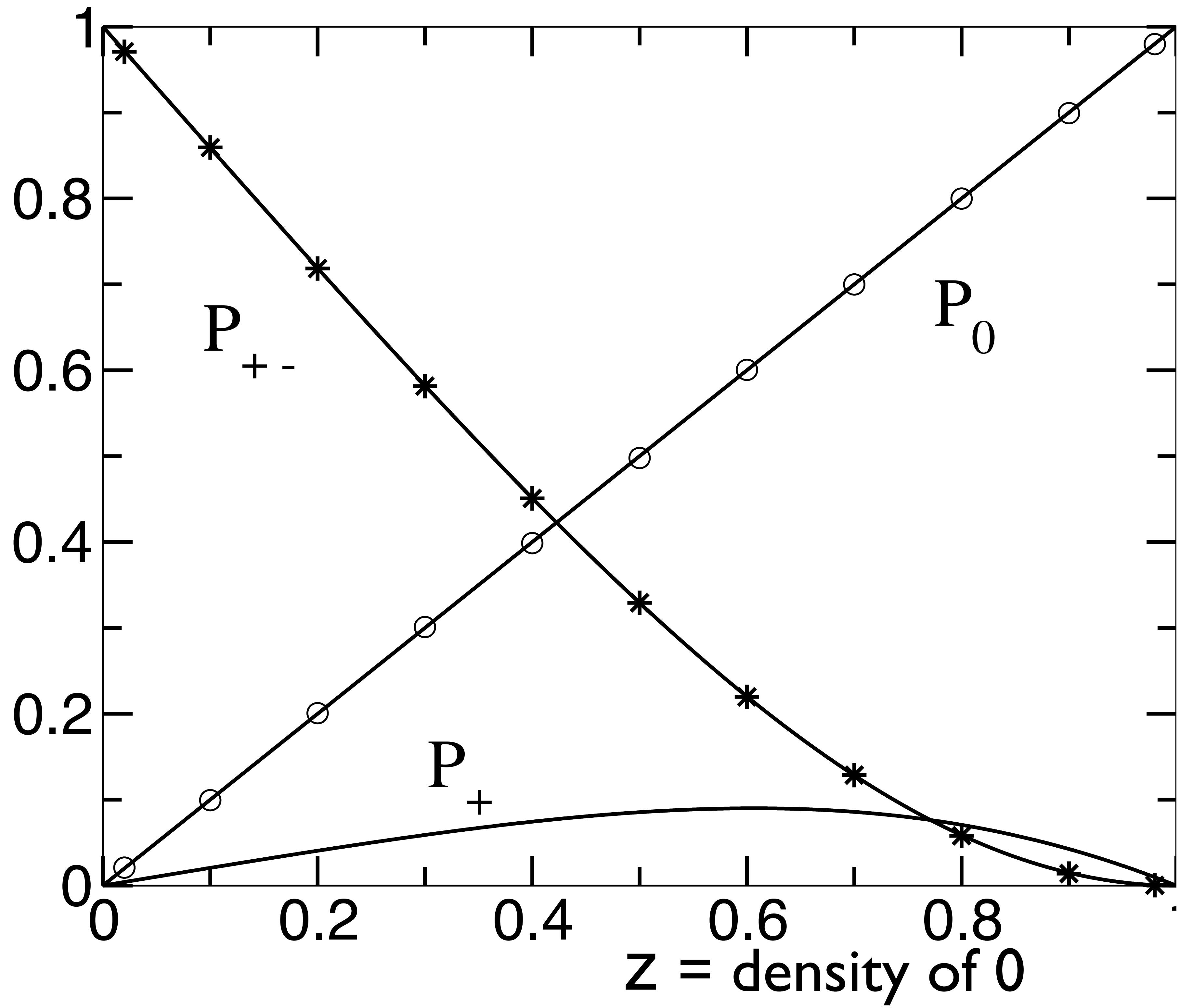
continuum limit:

$$
x\frac{\partial^2 F(x,y)}{\partial x^2} + y\frac{\partial^2 F(x,y)}{\partial y^2} = 0, \qquad
\begin{aligned}
F(x,0) &= 0 \\
F(0,y) &= 0 \\
F(x,1-x) &= 1
\end{aligned}
$$

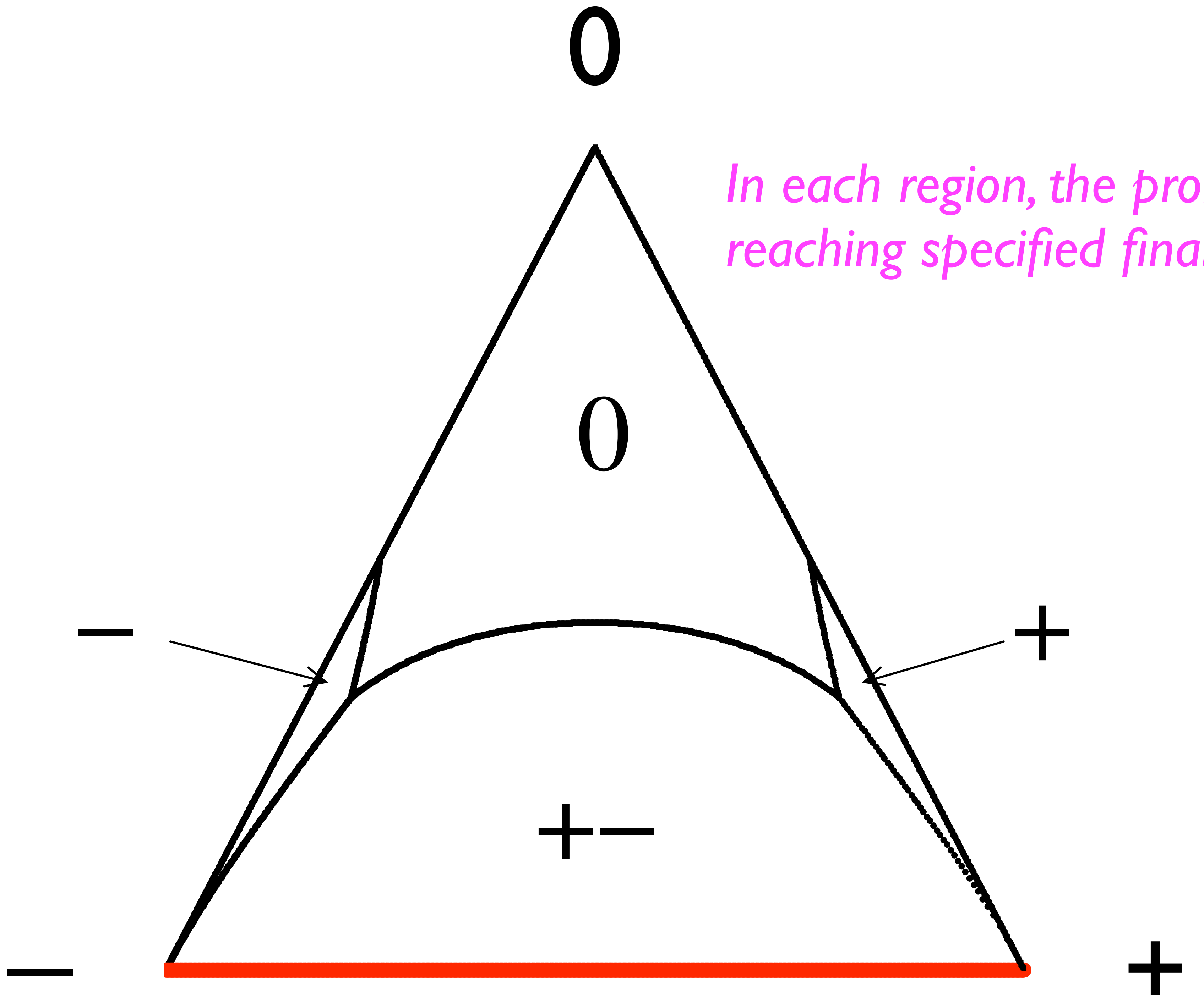
solution:

$$
F(x,y) = \sum_{n\ \text{odd}} \frac{2(2n+1)}{n(n+1)}\sqrt{xy}\,(x+y)^n\,P_n^1\left(\frac{x-y}{x+y}\right)
$$

Final state probabilities



Phase diagram



In each region, the probability of reaching specified final state is >50%

moral: *extremism promotes deadlock*

Outlook & some open questions

1. Heterogeneous voter model: fast consensus

   What is the route to consensus?

   Role of fluctuations?

   Behavior of the correlations?

2. Majority rule: complex relaxation to a simple final state

   Why do stripes occur?

   What is the critical dimension?

   What happens for more than 2 states?

3. Bounded compromise: rich bifurcation sequence

4. Spiteful extremists: deadlock & multiple final states

   Implications for real politics?