Capture of the Lamb: Diffusing Predators Seeking a Diffusing Prey

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We study the capture of a diffusing "lamb" by diffusing "lions" in one dimension. The capture dynamics is exactly soluble by probabilistic techniques when the number of lions is very small, and is tractable by extreme statistics considerations when the number of lions is very large. However, the exact solution for the general case of three or more lions is still not known.

I. INTRODUCTION

What is the survival probability of a diffusing lamb which is hunted by N hungry lions? Although this capture process is appealingly simple to define (see Fig. 1), its long-time behavior[1–3] poses a theoretical challenge because of the delicate interplay between the positions of the lamb and the closest lion. This model also illustrates a general feature of nonequilibrium statistical mechanics: life is richer in low dimensions. For spatial dimension $d > 2$, it is known that the capture process is "unsuccessful" (in the terminology of Ref. 1), as there is a nonzero probability for the lamb to survive to infinite time for any initial spatial distribution of the lions. This result is a consequence of the *transience* of diffusion for $d > 2$,[4,5] which means that two nearby diffusing particles in an unbounded $d > 2$ domain may never meet. For $d = 2$, capture is "successful", as the lamb dies with certainty. However, diffusing lions in $d = 2$ are such poor predators that the average lifetime of the lamb is infinite! Also, the lions are essentially independent,[1] so that the survival probability of a lamb in the presence of N lions in two dimensions is $S_N(t) \propto S_1(t)^N$, where $S_1(t)$, the survival probability of a lamb in the presence of a single lion, decays as[5] $(\ln t)^{-1}$.

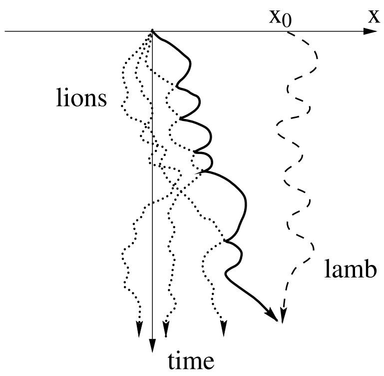

FIG. 1. Space-time evolution in one dimension of $N = 4$ diffusing lions (dotted lines) which all start at $x = 0$ and a single diffusing lamb (dashed) which starts at $x = x_0$. The trajectory of the closest ("last") lion, whose individual identity may change with time, is indicated by the heavy solid path.

Lions are more efficient predators in $d = 1$ because of the *recurrence* of diffusion,[4,5] which means that two diffusing particles are certain to meet eventually. The $d = 1$ case is also special because there are two distinct generic cases. When the lamb is surrounded by lions, the survival probability at a fixed time decreases rapidly with N because the safe zone which remains unvisited by lions at fixed time shrinks rapidly in N. This article focuses on the more interesting situation of N lions all to one side of the lamb (Fig. 1), for which the lamb survival probability decays as a power law in time with an exponent that grows only logarithmically in N.

We begin by considering a lamb and a single stationary lion in Section II. The survival probability of the lamb $S_1(t)$ is closely related to the first-passage probability of one-dimensional diffusion[4,5] and leads to $S_1(t) \sim t^{-1/2}$. It is also instructive to consider general lion and lamb diffusivities. We treat this two-particle system by mapping it onto an effective single-particle diffusion problem in two dimensions with an absorbing boundary to account for the death of the lamb when it meets the lion,[6] and then solving the two-dimensional problem by the image method. We apply this approach in Section III by mapping a diffusing lamb and two diffusing lions onto a single diffusing particle within an absorbing wedge whose opening angle depends on the particle diffusivities,[7] and then solving the diffusion problem in this absorbing wedge by classical methods.

In Section IV, we study $N \gg 1$ diffusing lions.[2,3] An essential feature of this system is that the motion of the closest ("last") lion to the lamb is biased towards the lamb, even though each lion diffuses isotropically. The many-particle system can be recast as a two-particle system consisting of the lamb and an absorbing boundary which, from extreme statistics,[8] moves to the right as $\sqrt{4D_L t \ln N}$, where D_L is the lion diffusivity. Because this time dependence matches that of the lamb's diffusion, the survival probability depends intimately on these two motions,[9–11] with the result that $S_N(t) \sim t^{-\beta_N}$ and $\beta_N \propto \ln N$. The logarithmic dependence of β_N on N reflects the fact that each additional lion poses a progressively smaller marginal peril to the lamb — it matters little whether the lamb is hunted by 99 or 100 lions. Amusingly, the value of β_N implies an infinite lamb lifetime for $N \leq 3$ and a finite lifetime otherwise. In the terminology of Ref. 1, the capture process changes from "successful" to "complete" when $N \geq 4$. We close with some suggestions for additional research on this topic.

II. SURVIVAL IN THE PRESENCE OF ONE LION

A. Stationary Lion and Diffusing Lamb

We begin by treating a lamb which starts at $x_0 > 0$ and a stationary lion at $x = 0$. In the continuum limit, the probability density $p(x,t)$ that the lamb is at any point $x > 0$ at time t satisfies the diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = D_\ell \frac{\partial^2 p(x,t)}{\partial x^2}, \tag{1}$$

where D_ℓ is the diffusivity (or diffusion coefficient). The probability density satisfies the boundary condition $p(x = 0, t) = 0$ to account for the death of the lamb if it reaches the lion at $x = 0$, and the initial condition $p(x, t = 0) = \delta(x - x_0)$. Equation (1) may be easily solved by the familiar image method.[5] For $x > 0$, $p(x,t)$ is the superposition of a Gaussian centered at x_0 and an "image" anti-Gaussian centered at $-x_0$:

$$p(x,t) = \frac{1}{\sqrt{4\pi D_\ell t}} \left[e^{-(x-x_0)^2/4D_\ell t} - e^{-(x+x_0)^2/4D_\ell t} \right]. \tag{2}$$

The image contribution ensures that the boundary condition at $x = 0$ is automatically satisfied, while the full solution satisfies both the initial condition and the diffusion equation. Thus Eq. (2) gives the probability density of the lamb for $x > 0$ in the presence of a stationary lion at $x = 0$.

The probability that the lamb meets the lion at time t equals the diffusive flux to $x = 0$ at time t. The flux is

$$F(t) = +D_\ell \frac{\partial p(x,t)}{\partial x}\bigg|_{x=0} = \frac{x_0}{\sqrt{4\pi D_\ell t^3}}\, e^{-x_0^2/4D_\ell t}. \tag{3}$$

The flux $F(t)$ is also the *first-passage probability* to the origin, namely, the probability that a diffusing lamb which starts at x_0 reaches $x = 0$ *for the first time* at time t. Note that in the long time limit, defined by $D_\ell t \gg x_0^2$, the first-passage probability reduces to $F(t) \to x_0/t^{3/2}$. This $t^{-3/2}$ time dependence is a characteristic feature of the first-passage probability in one dimension.[5]

The probability that the lamb dies by time t is the time integral of $F(t)$ up to time t. The survival probability is just the complementary fraction of these doomed lambs, that is,

$$\begin{aligned} S_1(t) &= 1 - \int_0^t F(t')\, dt', \\ &= 1 - \int_0^t \frac{x_0}{\sqrt{4\pi D_\ell t'^3}}\, e^{-x_0^2/4D_\ell t'}\, dt'. \end{aligned} \tag{4}$$

The integral in Eq. (4) can be reduced to a standard form by the substitution $u = x_0/\sqrt{4D_\ell t'}$ to give

$$S_1(t) = \mathrm{erf}\left(\frac{x_0}{\sqrt{4D_\ell t}}\right) \sim \frac{x_0}{\sqrt{\pi D_\ell t}} \quad \text{as } t \to \infty, \tag{5}$$

where $\mathrm{erf}(z) = (2/\sqrt{\pi})\int_0^z e^{-u^2}\, du$ is the error function.[12] The same expression for $S_1(t)$ can be obtained by integrating the spatial probability distribution in Eq. (2) over all $x > 0$.

An amusing feature of the $t^{-1/2}$ decay of the lamb survival probability is that although the lamb dies with certainty, its average lifetime, defined as $\langle t \rangle = \int_0^\infty t\, F(t)\, dt = \int_0^\infty S(t)\, dt \approx \int^\infty t^{-1/2}\, dt$, is infinite. This infinite lifetime arises because the small fraction of lambs which survive tend to move relatively far away from the lion. More precisely, the superposition of the Gaussian and anti-Gaussian in Eq. (2) leads to a lamb probability distribution which is peaked at a distance $(D_\ell t)^{1/2}$ from the origin, while its spatial integral decays as $(D_\ell t)^{-1/2}$.

B. Both Species Diffusing

What is the survival probability of the lamb when the lion also diffuses? In the rest frame of the lamb, the lion now moves if either a lion or a lamb hopping event occurs, and their separation diffuses with diffusivity equal to $D_\ell + D_L$ (see for example, Ref. 5), where D_L is the lion diffusivity. From the discussion of Section IIA, the lamb survival probability has the asymptotic time dependence $S_1(t) \sim x_0/\sqrt{\pi(D_\ell + D_L)t}$.

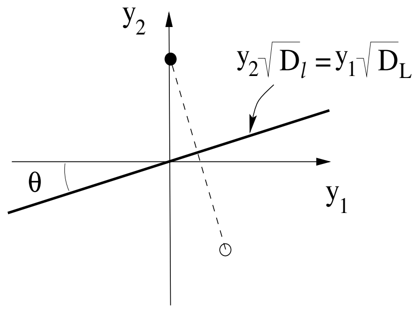

FIG. 2. Mapping of the lion and lamb coordinates in one dimension to the planar coordinates $y_1 = x_L/\sqrt{D_L}$ and $y_2 = x_\ell/\sqrt{D_\ell}$. The initial y-coordinates of the lion-lamb pair, $(0, \sqrt{D_\ell})$, and its image are indicated by the solid and open circles, respectively. Survival of the lamb, $y_1\sqrt{D_L} < y_2\sqrt{D_\ell}$, translates to the diffusing particle in the plane remaining above and to the left of the absorbing line $y_1\sqrt{D_L} = y_2\sqrt{D_\ell}$.

It is also instructive to determine the spatial probability distribution of the lamb. This distribution may be found conveniently by mapping the two-particle interacting system of lion at x_L and lamb at x_ℓ in one dimension to an effective single-particle system in two dimensions[6] and then applying the image method to solve the latter (see Fig. 2). To construct this mapping, we introduce the scaled coordinates $y_1 = x_L/\sqrt{D_L}$ and $y_2 = x_\ell/\sqrt{D_\ell}$ to render the two-dimensional diffusive trajectory (y_1, y_2) isotropic. The probability density in the plane, $p(y_1, y_2, t)$, must satisfy an absorbing boundary condition when $y_2\sqrt{D_\ell} = y_1\sqrt{D_L}$, corresponding to the death of the lamb when it meets the lion. For simplicity and without loss of generality, we assume that the lion and lamb are initially at $x_L(0) = 0$ and $x_\ell(0) = 1$ respectively, that is, $y_1(0) = 0$ and $y_2(0) = \sqrt{D_\ell}$. The probability density is therefore the sum of a Gaussian centered at $(y_1(0), y_2(0)) = (0, \sqrt{D_\ell})$ and an anti-Gaussian image. From the orientation of the absorbing boundary (Fig. 2), this image is centered at $(\sqrt{D_\ell}\sin 2\theta, -\sqrt{D_\ell}\cos 2\theta)$, where $\theta = \tan^{-1}\sqrt{D_L/D_\ell}$.

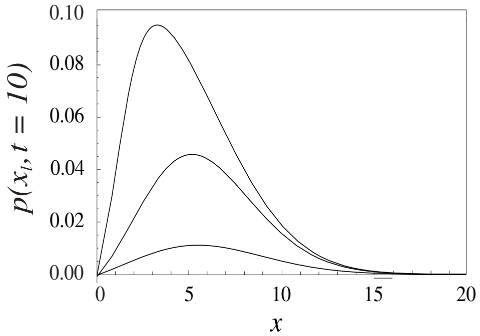

FIG. 3. Probability distribution of the lamb in one dimension at time $t = 10$ (Eq. (8)) when the lion and lamb are initially at $x_L = 0$ and $x_\ell = 1$, respectively. The cases shown are $r = D_\ell/D_L = 0.1$, 1, and 10 (bottom to top).

From this image representation, the probability density in two dimensions is

$$p(y_1, y_2, t) = \frac{1}{4\pi t}\left[e^{-[y_1^2 + (y_2 - \sqrt{D_\ell})^2]/4t} - e^{-[(y_1 - \sqrt{D_\ell}\sin 2\theta)^2 + (y_2 + \sqrt{D_\ell}\cos 2\theta)^2]/4t}\right]. \tag{6}$$

The probability density for the lamb to be at y_2 is the integral of the two-dimensional density over the accessible range of the lion coordinate y_1:

$$p(y_2,t) = \int_{-\infty}^{y_2 \cot\theta} p(y_1,y_2,t)\, dy_1 . \tag{7}$$

If we substitute the result (6) for $p(y_1,y_2,t)$, the integral in Eq. (7) can be expressed in terms of the error function. We then transform back to the original lamb coordinate $x_\ell = y_2\sqrt{D_\ell}$ by using $p(x_\ell,t)\,dx_\ell = p(y_2,t)\,dy_2$ to obtain

$$p(x_\ell,t) = \frac{1}{\sqrt{16\pi D_\ell t}}\left[e^{-(x_\ell-1)^2/4D_\ell t}\,\mathrm{erfc}\Big(-\frac{x_\ell\cot\theta}{\sqrt{4D_\ell t}}\Big) - e^{-(x_\ell+\cos 2\theta)^2/4D_\ell t}\,\mathrm{erfc}\Big(\frac{\sin 2\theta - x_\ell\cot\theta}{\sqrt{4D_\ell t}}\Big)\right], \tag{8}$$

where $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ is the complementary error function. A plot of $p(x_\ell,t)$ is shown in Fig. 3 for various values of the diffusivity ratio $r \equiv D_\ell/D_L$. The figure shows that the survival probability of the lamb rapidly decreases as the lion becomes more mobile. Note that when the lion is stationary, $\theta = 0$, and Eq. (8) reduces to Eq. (2).

III. TWO LIONS

To find the lamb survival probability in the presence of two diffusing lions, we generalize the above approach to map the three-particle interacting system in one dimension to an effective single diffusing particle in three dimensions with boundary conditions that reflect the death of the lamb whenever a lion is encountered.[7] Let us label the lions as particles 1 and 2, and the lamb as particle 3, with respective positions x_1, x_2, and x_3, and respective diffusivities D_i. It is again useful to introduce the scaled coordinates $y_i = x_i/\sqrt{D_i}$ which renders the diffusion in the y_i coordinates spatially isotropic. In terms of y_i, lamb survival corresponds to $y_2\sqrt{D_2} < y_3\sqrt{D_3}$ and $y_1\sqrt{D_1} < y_3\sqrt{D_3}$. These constraints mean that the effective particle in three-space remains behind the plane $y_2\sqrt{D_2} = y_3\sqrt{D_3}$ and to the left of the plane $y_1\sqrt{D_1} = y_3\sqrt{D_3}$ (Fig. 4(a)); this geometry is a wedge region of opening angle Θ defined by the intersection of these two planes. If the particle hits one of the planes, then one of the lions has killed the lamb.

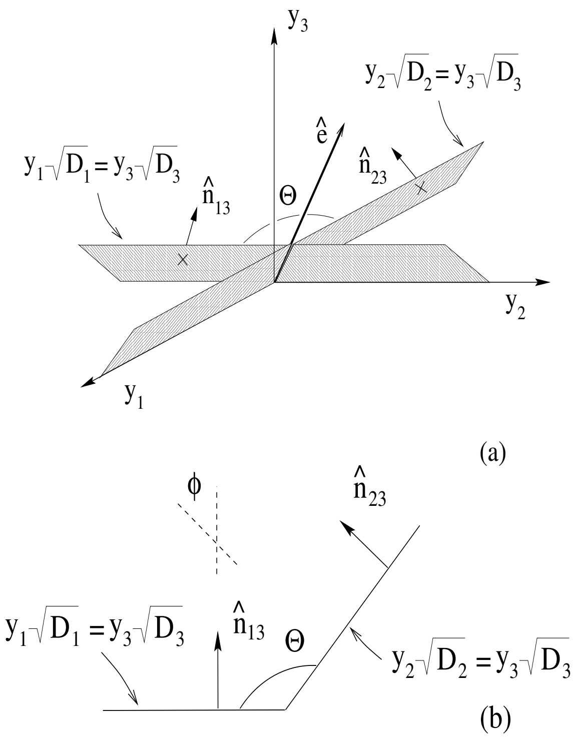

FIG. 4. (a) Mapping between the coordinates x_i of three diffusing particles on the line and a single isotropically diffusing particle in the three-dimensional space $y_i = x_i/\sqrt{D_i}$, subject to the constraints $y_1\sqrt{D_1} < y_3\sqrt{D_3}$ and $y_2\sqrt{D_2} < y_3\sqrt{D_3}$. The lamb survives if it remains within the wedge-shaped region of opening angle Θ. (b) Projection of the wedge onto a plane perpendicular to the $\hat{e}$ axis defined by the intersection of the two planes.

This mapping therefore provides the lamb survival probability, since it is known that the survival probability of a diffusing particle within this absorbing wedge asymptotically decays as[13]

$$S_{\rm wedge}(t) \sim t^{-\pi/2\Theta}. \tag{9}$$

For completeness, we derive this asymptotic behavior by mapping the diffusive system onto a corresponding electrostatic system in Appendix A. To determine the value of Θ which corresponds to our 3-particle system, notice that the unit normals to the planes $y_1\sqrt{D_1} = y_3\sqrt{D_3}$ and $y_2\sqrt{D_2} = y_3\sqrt{D_3}$ are $\hat{\mathbf{n}}_{13} = (-\sqrt{D_1}, 0, \sqrt{D_3})/\sqrt{D_1 + D_3}$ and $\hat{\mathbf{n}}_{23} = (0, -\sqrt{D_2}, \sqrt{D_3})/\sqrt{D_2 + D_3}$, respectively. Consequently $\cos\phi = \hat{\mathbf{n}}_{13} \cdot \hat{\mathbf{n}}_{23}$ (Fig. 4(b)), and the wedge angle is $\Theta = \pi - \phi = \pi - \cos^{-1}[D_3/\sqrt{(D_1 + D_3)(D_2 + D_3)}]$. If we take $D_1 = D_2 = D_L$ for identical lions, and $D_3 = D_\ell$, the survival exponent for the lamb is

$$\beta_2(r) = \frac{\pi}{2\Theta} = \left[2 - \frac{2}{\pi}\cos^{-1}\frac{r}{1+r}\right]^{-1}, \tag{10}$$

where $r = D_\ell/D_L$.

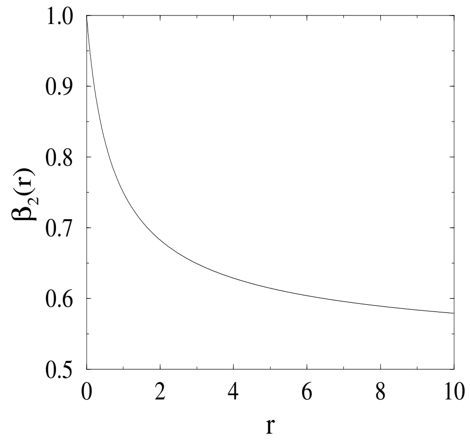

FIG. 5. The survival exponent $\beta_2(r)$ given by Eq. (10) versus the diffusivity ratio r.

The dependence of $\beta_2(r)$ on the diffusivity ratio r is shown in Fig. 5. This exponent monotonically decreases from 1 at $r = 0$ to 1/2 for $r \to \infty$. The former case corresponds to a stationary lamb, where the two lions are statistically independent and $S_2(t) = S_1(t)^2$. On the other hand, when $r \to \infty$ the lamb diffuses rapidly and the motion of the lions becomes irrelevant. This limit therefore reduces to the diffusion of a lamb and a stationary absorber, for which $S_2(t) = S_1(t)$. Finally, for $D_\ell = D_L$, $\beta_2 = 3/4 < 2\beta_1$, and equivalently, $S_2(t) > S_1(t)^2$. This inequality reflects the fact that the incremental threat to the lamb from the second lion is less than the first.

IV. MANY LIONS

The above construction can, in principle, be extended by recasting the survival of a lamb in the presence of N lions as the survival of a diffusing particle in $N+1$ dimensions within an absorbing hyper-wedge defined by the intersection of the N half-spaces $x_i < x_{N+1}$, $i = 1, 2, \ldots, N$. This approach has not led to a tractable analytical solution. On the other hand, numerical simulations[1] indicate that the exponent β_N grows slowly with N, with $\beta_3 \approx 0.91$, $\beta_4 \approx 1.03$, and $\beta_{10} \approx 1.4$. The understanding of the slow dependence of β_N on N is the focus of this section.

A. LOCATION OF THE LAST LION

One way to understand the behavior of the survival probability is to focus on the lion closest to the lamb, because this last lion ultimately kills the lamb. As was shown in Fig. 1, the individual identity of this last lion can change with time due to the crossing of different lion trajectories. In particular, crossings between the last lion and its left neighbor lead to a systematic rightward bias of the last lion. This bias is stronger for increasing N, due to the larger number of crossings of the last lion, and this high crossing rate also leads to $x_{\rm last}(t)$ becoming smoother as N increases (Fig. 6). This approach of the last lion to the lamb is the mechanism which leads to the survival probability of the lamb decaying as $t^{-\beta_N}$, with β_N a slowly increasing function of N.

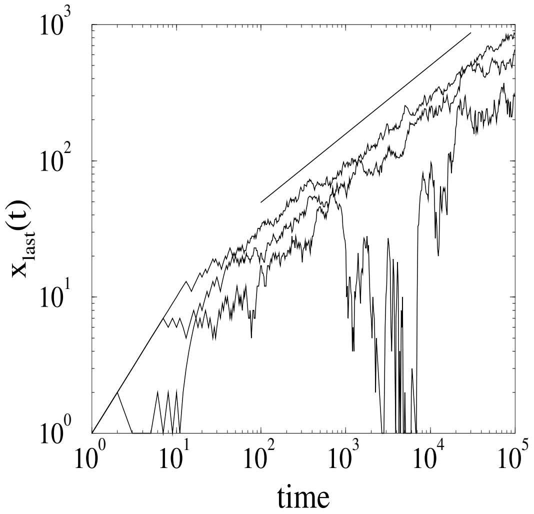

FIG. 6. Time dependence of $x_{\rm last}$ for a single realization of $N = 4$, 64, and 1028 lions (bottom to top). This data was generated by tracking the position of the rightmost among N lions, each of which performs a nearest-neighbor discrete-time random walk starting from $x = 0$. The coincidence of the data and the linear early-time growth of $x_{\rm lst}$ are artifacts of the discrete random walk motion. The straight line of slope 1/2 indicates the expected long-time behavior.

To determine the properties of this last lion, suppose that $N \gg 1$ lions are initially at the origin. If the lions perform nearest-neighbor, discrete-time random walks, then at short times, $x_{\rm last}(t) = t$. This trivial dependence persists as long as the number of lions at the last site in their spatial distribution is much greater than one. In this case there is a large probability that one of these lions will hop to the right, thus maintaining the deterministic growth of $x_{\rm last}$. This growth will continue as long as $\frac{N}{\sqrt{4\pi D_L t}} e^{-t^2/4D_L t} \gg 1$, that is, for $t \ll 4D_L \ln N$. At long times, an estimate for the location of the last lion is provided by the condition[8]

$$\int_{x_{\rm last}}^{\infty} \frac{N}{\sqrt{4\pi D_L t}}\, e^{-x^2/4D_L t}\, dx = 1. \tag{11}$$

Equation (11) specifies that there is one lion out of an initial group of N lions which is in the range $[x_{\rm last}, \infty]$. Although the integral in Eq. (11) can be expressed in terms of the complementary error function, it is instructive to evaluate it explicitly by writing $x = x_{\rm last} + \epsilon$ and re-expressing the integrand in terms of ϵ. We find that

$$\int_{x_{\rm last}}^{\infty} \frac{N}{\sqrt{4\pi D_L t}}\, e^{-x_{\rm last}^2/4D_L t}\, e^{-x_{\rm last}\epsilon/2D_L t}\, e^{-\epsilon^2/4D_L t}\, d\epsilon = 1. \tag{12}$$

Over the range of ϵ for which the second exponential factor is non-negligible, the third exponential factor is nearly equal to unity. The integral in Eq. (12) thus reduces to an elementary form, with the result

$$\frac{N}{\sqrt{4\pi D_L t}}\, e^{-x_{\rm last}^2/4D_L t}\, \frac{2D_L t}{x_{\rm last}} = 1. \tag{13}$$

If we define $y = x_{\rm last}/\sqrt{4D_L t}$ and $M = N/\sqrt{4\pi}$, the condition in Eq. (13) can be simplified to

$$y e^{y^2} = M, \tag{14}$$

with the solution

$$y = \sqrt{\ln M}\left(1 - \frac{1}{4}\frac{\ln(\ln M)}{\ln M} + \ldots\right). \tag{15}$$

In addition to obtaining the mean location of the last lion, extreme statistics can be used to find the spatial probability of the last lion. For completeness, this calculation is presented in Appendix B.

To lowest order, Eq. (15) gives

$$x_{\rm last}(t) \sim \sqrt{4D_L \ln N\, t} \equiv \sqrt{A_N t}, \tag{16}$$

for finite N. For $N = \infty$, $x_{\rm last}(t)$ would always equal t if an infinite number of discrete random walk lions were initially at the origin. A more suitable initial condition therefore is a concentration c_0 of lions uniformly distributed from $-\infty$ to 0. In this case, only $N \propto \sqrt{c_0^2 D_L t}$ of the lions are "dangerous," that is, within a diffusion distance from the edge of the pack and thus potential candidates for killing the lamb. Consequently, for $N \to \infty$, the leading behavior of $x_{\rm last}(t)$ becomes

$$x_{\rm last}(t) \sim \sqrt{2D_L \ln(c_0^2 D_L t)\, t}\,. \tag{17}$$

As we discuss in Section IVB, the survival probability of the lamb in the presence of many lions is essentially determined by this behavior of $x_{\rm last}$.

B. LAMB SURVIVAL PROBABILITY FOR LARGE N

An important feature of the time dependence of $x_{\rm last}$ is that fluctuations decrease for large N (Fig. 6). Therefore the lamb and N diffusing lions can be recast as a two-body system of a lamb and an absorbing boundary which *deterministically* advances toward the lamb as $x_{\rm last}(t) = \sqrt{A_N t}$.

To solve this two-body problem, it is convenient to change coordinates from $[x, t]$ to $[x' = x - x_{\rm last}(t), t]$ to fix the absorbing boundary at the origin. By this construction, the diffusion equation for the lamb probability distribution is transformed to the convection-diffusion equation

$$\frac{\partial p(x',t)}{\partial t} - \frac{x_{\rm last}}{2t}\frac{\partial p(x',t)}{\partial x'} = D_\ell \frac{\partial^2 p(x',t)}{\partial x'^2}, \quad (0 \le x' < \infty) \tag{18}$$

with the absorbing boundary condition $p(x' = 0, t) = 0$. In this reference frame which is fixed on the average position of the last lion, the second term in Eq. (18) accounts for the bias of the lamb towards the absorber with a "velocity" $-x_{\rm last}/2t$. Because $x_{\rm last} \sim \sqrt{A_N t}$ and $x' \sim \sqrt{D_\ell t})$ have the same time dependence, the lamb survival probability acquires a nontrivial dependence on the dimensionless parameter A_N/D_ℓ.[9–11] Such a dependence is in contrast to the cases $x_{\rm last} \ll x'$ or $x_{\rm last} \gg x'$, where the asymptotic time dependence of the lamb survival is controlled by the faster of these two co-ordinates. Such a phenomenon occurs whenever there is a coincidence of fundamental length scales in the system (see, for example, Ref. 14).

Equation (18) can be transformed into the parabolic cylinder equation by the following steps.[3] First introduce the dimensionless length $\xi = x'/x_{\rm last}$ and make the following scaling ansatz for the lamb probability density,

$$p(x',t) \sim t^{-\beta_N - 1/2} F(\xi). \tag{19}$$

The power law prefactor in Eq. (19) ensures that the integral of $p(x', t)$ over all space, namely the survival probability, decays as $t^{-\beta_N}$, and $F(\xi)$ expresses the spatial dependence of the lamb probability distribution in scaled length units. This ansatz codifies the fact that the probability density is not a function of x' and t separately, but is a function only of the dimensionless ratio $x'/x_{\rm last}(t)$. The scaling ansatz provides a simple but powerful approach for reducing the complexity of a wide class of systems with a divergent characteristic length scale as $t \to \infty$.[14]

If we substitute Eq. (19) into Eq. (18), we obtain

$$\frac{D_\ell}{A_N}\frac{d^2F}{d\xi^2} + \frac{1}{2}(\xi+1)\frac{dF}{d\xi} + \big(\beta_N + \frac{1}{2}\big)F = 0. \tag{20}$$

Now introduce $\eta = (\xi + 1)\sqrt{A_N/2D_\ell}$ and $F(\xi) = e^{-\eta^2/4}\,\mathcal{D}(\eta)$ in Eq. (20). This substitution leads to the parabolic cylinder equation of order $2\beta_N$[15]

$$\frac{d^2\mathcal{D}_{2\beta_N}}{d\eta^2} + \left[2\beta_N + \frac{1}{2} - \frac{\eta^2}{4}\right]\mathcal{D}_{2\beta_N} = 0, \tag{21}$$

subject to the boundary condition, $\mathcal{D}_{2\beta_N}(\eta) = 0$ for both $\eta = \sqrt{A_N/2D_\ell}$ and $\eta = \infty$. Equation (21) has the form of a Schrödinger equation for a quantum particle of energy $2\beta_N + \frac{1}{2}$ in a harmonic oscillator potential $\eta^2/4$ for

$\eta > \sqrt{A_N/2D_\ell}$, but with an infinite barrier at $\eta = \sqrt{A_N/D_\ell}$.[16] For the long time behavior, we want the ground state energy in this potential. For $N \gg 1$, we may approximate this energy as the potential at the classical turning point, that is, $2\beta_N + \frac{1}{2} \simeq \eta^2/4$. We therefore obtain $\beta_N \sim A_N/16D_\ell$. Using the value of A_N given in Eqs. (16) and (17) gives the decay exponent

$$\beta_N \sim \begin{cases} \frac{D_L}{4D_\ell} \ln N, & N \text{ finite} \\ \\ \frac{D_L}{8D_\ell} \ln t. & N = \infty \end{cases} \tag{22}$$

The latter dependence of β_N implies that for $N \to \infty$, the survival probability has the log-normal form

$$S_\infty(t) \sim \exp\left(-\frac{D_L}{8D_\ell} \ln^2 t\right). \tag{23}$$

Although we obtained the survival probability exponent β_N for arbitrary diffusivity ratio $r = D_\ell/D_L$, simple considerations give different behavior for $r \gg 1$ and $r \ll 1$. For example, for $r = 0$ (stationary lamb) the survival probability decays as $t^{-N/2}$. Therefore Eq. (22) can no longer apply for $r < N^{-1}$, where $\beta_N(r)$ becomes of order N. Conversely, for $r = \infty$ (stationary lions), the survival probability of the lamb decays as $t^{-1/2}$. Thus Eq. (22) will again cease to be valid for $r > \ln N$, where $\beta_N(r)$ becomes of order unity. By accounting for these limits,[3] the dependence of β_N on the diffusivity ratio r is expected to be

$$\beta_N(r) = \begin{cases} N/2 & r \ll 1/N \\ (1/4r)\ln(4N) & 1/N \ll r \ll \ln N \\ 1/2 & r \gg \ln N. \end{cases} \tag{24}$$

The r dependence of β_N in the intermediate regime of $1/N \ll r \ll \ln N$ generalizes the exponents given in Eq. (10) for the three-particle system to general N.

V. DISCUSSION

We investigated diffusive capture of a lamb in one dimension by using of several essential techniques of nonequilibrium statistical mechanics including first-passage properties of diffusion, extreme value statistics, electrostatic analogies, scaling analysis, and moving boundary value theory. These tools provide an appealing physical description for the survival probability of the lamb in the presence of N lions for the cases of very small and very large N. Nevertheless, the exact solution to capture of the lamb remains elusive for $N \geq 3$.

We close by suggesting several avenues for further study:

1. *Better Simulation Methods.* Previous simulations of this system[1] followed the random walk motion of one lamb and N lions until the lamb was killed. This type of simulation is simple to construct. One merely places the lamb and lions on a one-dimensional lattice and have them perform independent nearest-neighbor random walks until one lion lands on the same site as the lamb. The survival probability is obtained by averaging over a large number of realizations of this process. However, following the motion of discrete random walks is inefficient, because it is unlikely for the lamb to survive to long times and many realizations of the process must be simulated to obtain accurate long-time data. In principle, a much better approach would be to propagate the exact probability distribution of the particles in the system.[17] Can such an approach be developed for the lamb-lion capture process? Another possibility is to devise a simple discrete random-walk process to simulate the motion of the last lion. Such a construction would permit consideration of just the lamb and the last lion, thus providing significant computational efficiency.

2. *The Last Lion.* Extreme value statistics provides the spatial probability distribution of the last lion. We may also ask other basic questions: How long is a given lion the "last" one? How many lead changes of the last lion occur up to time t? How many different lions may be in the lead up to time t? What is the probability that a particular lion is never in the lead? Methods to investigate some of these issues are also outlined in the article by Schmittmann and Zia[18] in this journal issue.

3. *Spatial Probability Distribution of the Lamb.* As we have seen for the case of one lion, the spatial distribution of the lamb is a useful characteristic of the capture process. What happens for large N? In principle, this information is contained in the solution to the parabolic cylinder equation for the scaled probability distribution (Eq. (21)). The most interesting behavior is the form of the distribution close to the absorbing boundary, where

the interaction between the lions and the lamb is strongest. For $N = 1$ lion, this distribution decays linearly to zero as a function of the distance to the lion, while for $N = 2$, the distribution has a power law decay in the distance to the last lion which depends on the diffusivity ratio D_ℓ/D_L. What happens for general N and for general diffusivity ratio? Is there a physical way to determine this behavior?

4. *Two-Sided Problem.* If N lions are located on both sides of the lamb, then the lamb is relatively short-lived because there is no escape route. One can again construct a mapping between the $N+1$-particle reacting system and the diffusion of an effective particle in an absorbing wedge-shaped domain in $N+1$ dimensions. From this mapping, $S_N(t)$ decays as $t^{-\gamma_N}$, but the dependence of γ_N on N and diffusivity ratio is unknown. It is clear, however, that the optimal strategy for the surrounded lamb is to remain still, in which case the lions are statistically independent and we then recover $S_N(t) \sim t^{-N/2}$. Is there a simple approach that provides the dependence of γ_N on N for arbitrary diffusivity ratio? Finally, $S_\infty(t)$ exhibits a stretched exponential decay in time, $\exp(-t^{1/2})$.[19,20] What is that nature of the transition from finite N to infinite N behavior?

5. *Intelligent Predators and Prey.* In a more realistic capture process, lions would chase the lamb, while the lamb would attempt to run away. What are physically-reasonable and analytically-tractable rules for such directed motion which would lead to new and interesting kinetic behaviors?

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APPENDIX A: SURVIVAL PROBABILITY OF A DIFFUSING PARTICLE WITHIN A WEDGE

The survival probability of a diffusing particle within an absorbing wedge can be derived by solving the diffusion equation in this geometry.[7,13] We provide an alternative derivation of this result can also be obtained by developing a correspondence between diffusion and electrostatics in the same boundary geometry. Although the logic underlying the correspondence is subtle, the result is simple and has wide applicability.

The correspondence rests on the fact that the integral of the diffusion equation over all time reduces to the Poisson equation. This time integral is

$$\int_0^\infty \left\{ D\nabla^2 p(r,\theta,t) = \frac{\partial p(r,\theta,t)}{\partial t} \right\} dt. \tag{25}$$

If one defines an electrostatic potential by $\Phi(r,\theta) = \int_0^\infty p(r,\theta,t)\,dt$, Eq. (25) can be written as

$$\nabla^2\Phi(r,\theta) = -\frac{1}{D}\left[p(r,\theta,t=\infty) - p(r,\theta,t=0)\right]. \tag{26}$$

For a boundary geometry such that the asymptotic survival probability in the diffusive system is zero, then Eq. (26) is just the Poisson equation, with the initial condition in the diffusive system corresponding to the charge distribution in the electrostatic system, and with absorbing boundaries in the diffusive system corresponding to grounded conductors.

To exploit this analogy, we first note that the electrostatic potential in the wedge decays as $\Phi(r,\theta) \sim r^{-\pi/\Theta}$ for $r \to \infty$, for any localized charge distribution.[21] Because the survival probability of a diffusing particle in the absorbing wedge is given by $S(t) = \int p(r,\theta,t)\,dA$, where the integral is over the area of the wedge, we find the following basic relation between $S(t)$ and the electrostatic potential in the same boundary geometry

$$\begin{aligned}
\int_0^t S(t)\,dt &= \int_0^t dt \int p(r,\theta,t)\,dA \\
&\approx \int_0^{\sqrt{Dt}} r\,dr \int_0^\Theta d\theta\ \Phi(r,\theta) \\
&\sim \int_0^{\sqrt{Dt}} r^{1-\pi/\Theta}\,dr \\
&\propto t^{1-\pi/2\Theta}.
\end{aligned} \tag{27}$$

In evaluating the time integral of the survival probability, we use the fact that particles have time to diffuse to radial distance $\sqrt{Dt}$ but no further. Thus in the second line of Eq. (27), the time integral of the probability distribution reduces to the electrostatic potential for $r < \sqrt{Dt}$ and is essentially zero for $r > \sqrt{Dt}$. Finally, by differentiating the last equality in Eq. (27) with respect to time we recover Eq. (9).

APPENDIX B: SPATIAL DISTRIBUTION OF THE LAST LION BY EXTREME STATISTICS

It is instructive to apply extreme statistics to determine the probability distribution for the location of the last lion from an ensemble of N lions.[8] Let $p(x) = \frac{1}{\sqrt{4\pi D_L t}} e^{-x^2/4D_L t}$ be the (Gaussian) probability distribution of a single lion. Then $p_>(x) \equiv \int_x^\infty p(x')\,dx'$ is the probability that a diffusing lion is in the range $[x, \infty]$ and similarly $p_<(x) = 1 - p_>(x)$ is the probability that the lion is in the range $[-\infty, x]$. Let $L_N(x)$ be the probability that the last lion out of a group of N is located at x. This extremal probability is given by

$$L_N(x) = Np(x)p_<(x)^{N-1}. \tag{28}$$

That is, one of the N lions is at x, while the remaining $N-1$ lions are in the range $[-\infty, x]$. If we evaluate the factors in Eq. (28), we obtain a double exponential distribution:[8]

$$\begin{aligned} L_N(x) &= \frac{N}{\sqrt{4\pi D_L t}} e^{-x^2/4D_L t} \left[1 - \int_x^\infty \frac{1}{\sqrt{4\pi D_L t}} e^{-x^2/4D_L t}\,dx\right]^{N-1}, \\ &\sim \frac{N}{\sqrt{4\pi D_L t}} e^{-x^2/4D_L t} \exp\left[-\frac{N-1}{\sqrt{4\pi D_L t}} \int_x^\infty e^{-x^2/4D_L t}\,dx\right]. \end{aligned} \tag{29}$$

When N is large, then $x/\sqrt{4D_L t}$ is also large, and we can asymptotically evaluate the integral in the exponential in Eq. (29). Following Eq. (14), it is convenient to express the probability distribution in terms of $M = N/\sqrt{4\pi}$ and $y = x_{\rm last}/\sqrt{4D_L t}$. If we use $L_N(y)\,dy = L_N(x)\,dx$, we obtain

$$L_N(y) \simeq 2Me^{-y^2} \exp(-Me^{-y^2}/y). \tag{30}$$

The most probable value of $x_{\rm last}$ is determined by the requirement that $L'_N(y) = 0$. This condition reproduces $ye^{y^2} = M$ given in Eq. (14).

We may also estimate the width of the distribution from its inflection points, that is, when $L''_M(y) = 0$. By straightforward calculation, $L''_N(y) = 0$ at

$$y_\pm \simeq \sqrt{\ln(M/k_\pm)} \approx \sqrt{\ln M}\Big(1 - \frac{\ln k_\pm}{\ln M}\Big), \tag{31}$$

where $k_\pm = (3 \pm \sqrt{5})/2$. Therefore as $N \to \infty$, the width of $L_N(y)$ vanishes as $1/\sqrt{\ln M}$. This behavior is qualitatively illustrated in Fig. 6, where the fluctuations in $x_{\rm last}(t)$ decrease dramatically as N increases. This decrease can also be understood from the form of the extreme distribution $L_N(x)$ in Eq. (29). The large-x decay of $L_N(x)$ is governed by $p(x)$, while the double exponential factor becomes an increasingly sharp step at $x_{\rm step} \approx \sqrt{4D_L t \ln N}$ as N increases. The product of these two factors leads to $L_N(x)$ essentially coinciding with $p(x)$ for $x > x_{\rm step}$ and $L_N(x) \approx 0$ for $x < x_{\rm step}$.

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