Dynamics of Strategic Three-Choice Voting

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In certain parliamentary democracies, there are two major parties that move in and out of power every few elections, and a third minority party that essentially never governs. We present a simple model to account for this phenomenon, in which minority party supporters sometimes vote ideologically (for their party) and sometimes strategically (against the party they like the least). The competition between these disparate tendencies reproduces the empirical observation of two parties that frequently exchange majority status and a third party that is almost always in the minority.

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I. INTRODUCTION

A feature of governance in several countries with parliamentary elections—typically British Commonwealth countries and Britain itself—is that two major parties move in and out of power every few elections, while a smaller third party either has never or rarely governed. This lack of representation of the minority party occurs even though its vote fraction can be close to that of the major parties.

Governance is determined by the party (or coalition) that has the majority of members of parliament (MPs). Each MP is the candidate with the most votes in each parliamentary district election. This voting system makes it difficult for a minor party to gain representation that mirrors its vote fraction. For example, if one party (out of three) receives 30% of the vote in every district while the other two parties equally share the remaining 70%, then the minority party will have no MPs although it is supported by nearly 1 in 3 voters. As an illustration [1], in the 1983 British election, the Conservative party won 42.4% of the popular vote and 397 seats (61.1% of 650 seats), the Labor party won 27.6% of the vote and 209 seats (32.2% of 650), while the traditional third-place Liberal party won 25.4% of the vote but only 23 seats (3.56% of 650). Similarly, Canada had two major parties for much of the 20^{th} century—Liberals (center-left), Progressive Conservatives (center-right), and a smaller, but still national scale, New Democratic Party (leftist) that has never governed [2, 3].

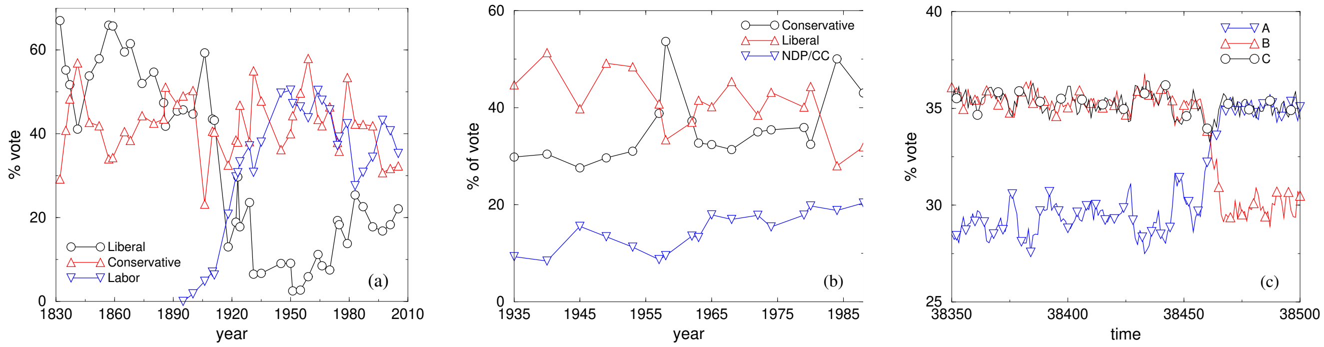

FIG. 1: (a) British election results since 1830. (b) Canadian election results from 1935–90; here NDP/CC denotes the New Democratic Party and its Cooperative Commonwealth party antecedent. (c) Representative simulation results for the time evolution of the densities of the three states in the strategic voter model with state-dependent strategic bias for 50000 agents and parameter $x_0 = 0.104$ (see text for definition of x_0).

To understand how such a voting pattern can arise, consider the case of the traditional minority Liberal party in

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Britain. If their election fortunes seem promising, then supporters are likely to be galvanized to vote for their party. However, if Liberal prospects seem bleak and the diametrically opposed Conservative party appears strong, a Liberal may vote for the Labor party to forestall the Conservatives. This strategy clearly disfavors the minority party (Fig. 1). In fact, when the number of parties is larger than two, ambiguous outcomes for the voting preference, such as the Condorcet paradox [4], can easily arise. There is also a richer range of phenomenology than in two-party voting [5].

To account for the endemic weakness of a minority party, we introduce the "strategic" voter model that encapsulates the strategic/ideological dichotomy outlined above. The original voter model [6] is a paradigmatic non-equilibrium process that describes ordering in non-equilibrium systems and consensus in opinion dynamics [7]. In the voter model, agents are endowed with a 2-state opinion variable. The dynamics is defined by picking a voter at random and updating its opinion and repeating *ad infinitum.* In the update event, the agent adopts the opinion of a random neighbor. Neighbors can be defined on a complete graph (interaction with any other agent equiprobably), across the links of regular lattices [8] or complex networks [9], or on adaptive graphs [10]. Various extensions of the dynamics itself have been considered [11].

In the next section, we define two natural versions of strategic voting and determine their dynamics in the mean-field limit. Typically, the population is driven to a steady state where a single party is perpetually in the minority unless the strategic bias is extremely weak. In the symmetric-breaking steady state, discrete fluctuations ultimately cause the minority party to escape its minority status on a slow time scale. It appears possible that election results (Fig. 1) would exhibit these same oscillations if the data continued for a much longer time period.

II. STRATEGIC VOTING

We consider a population of N voters on a complete graph that have three possible opinions states: A, B and C. The idealized complete graph is used because of its simplicity and analytical tractability. Each of these states is equivalent, in contrast to real political parties, and our model thus does not incorporate any subjective political attributes. We denote the fractions of the three types of voters as a, b, and c. The state of each individual evolves by the following voter-like dynamics:

- With rate T, a voter spontaneously changes to another state equiprobably. This thermalization step ensures that consensus is never reached.
- With rate r_{ij}, a random pair of voters in states i and j is picked and one member of this pair adopts the state of the other voter. In general, r_{ij} does not equal r_{ji} (see below).

In the mean-field limit ($N \to \infty$), the fractions of each voter species evolve according to the rate equations:

$$\begin{aligned}
\dot{a} &= T(b + c - 2a) + r_{AC}\, ac + r_{AB}\, ab \\
\dot{b} &= T(c + a - 2b) + r_{BA}\, ba + r_{BC}\, bc \\
\dot{c} &= T(a + b - 2c) + r_{CA}\, ca + r_{CB}\, cb\,.
\end{aligned} \tag{1}$$

The terms proportional to T account for spontaneous opinion changes, while the remaining terms account for voter-like updating. In the classic voter model [6, 15], each $r_{\alpha\beta} = 0$, because a voter pair $\alpha\beta$ can change to $\alpha\alpha$ or to $\beta\beta$ equiprobably; thus the density of each species is conserved. In our model, we only require $r_{\alpha\beta} + r_{\beta\alpha} = 0$ to conserve the total density.

For the strategic voter model, we take $r_{\alpha\beta}$ to be nonzero when one of α or β denotes a minority state. A simple choice is a strategic bias that is independent of the fractions of each species, *viz*:

$$r_{AB} = -r_{BA} = \begin{cases} +r & B \text{ minority}, \\ \ \ 0 & C \text{ minority}, \\ -r & A \text{ minority}, \end{cases}$$

(and cyclic permutations for r_{AC} and r_{BC}). Thus if B is in the minority, then an AB interaction favors the outcome AA rather than BB because $r_{AB} > 0$ and $r_{BA} < 0$. Conversely, if A is in the minority, an AB interaction favors BB rather than AA. If neither A or B are in the minority, then they undergo conventional voter dynamics in which their average densities do not change. While it would be politically more realistic to have non-symmetric interactions in which the minority species has a definite preference for one of the two non-minority species, this more complicated model does not yield any additional insights about strategic voting.

In the $c_<$ sector of the composition triangle (where c is in the minority, see Fig. 2), the rate equations reduce to:

$$\begin{aligned}\dot{a} &= T(1-3a) + rac\\ \dot{b} &= T(1-3b) + rbc\\ \dot{c} &= T(1-3c) - rc(1-c).\end{aligned} \tag{2}$$

Here we use $a+b+c=1$ to simplify the T-dependent terms and the last term in the equation for $\dot{c}$. Similar equations hold for the $a_<$ and $b_<$ sectors by cyclic permutations.

We will also study a more realistic strategic voting in which the strategic bias vanishes as the minority population approaches those of the other two states. In terms of election sentiment, if a minority party supporter believes that his/her party will win in an upcoming election, then there is every reason to vote ideologically with the party and not strategically. For simplicity we consider the case where the strategic bias varies linearly in the depth of minority status as quantified by the density-dependent strategic voting rate $r = r_0[(a+b)/2 - c]$ in Eqs. (2) (the $c_<$ sector), and analogously for the other sectors of the composition triangle.

III. SOLUTION TO THE RATE EQUATIONS

A. Constant Strategic Bias

For a strategic bias with a fixed rate r, we solve the last of Eqs. (2) by rewriting it in the factorized form

$$\frac{1}{r}\frac{dc}{dt} = c^2 - c\,(1+3x) + x \equiv (c-c_+)(c-c_-). \tag{3}$$

Here $x \equiv \frac{T}{r}$ quantifies the relative importance of the strategic bias (with strong bias corresponding to small x), and

$$c_\pm = \frac{1}{2}\left[(1+3x) \pm \sqrt{1+2x+9x^2}\right] \equiv \frac{1}{2}\left[(1+3x) \pm \frac{1}{\tau}\right].$$

We now apply a partial fraction expansion to Eq. (3) to render it integrable by elementary means and the solution is

$$c(t) = \frac{c_- - c_+\,\mathcal{C}\,e^{-rt/\tau}}{1-\mathcal{C}\,e^{-rt/\tau}}, \qquad \text{where} \qquad \mathcal{C} = \frac{c(0)-c_-}{c(0)-c_+}\ . \tag{4}$$

Thus $c(t)$ converges to its steady-state value of $c^* = c_-$ exponentially in time. The value of c^* sharply goes to zero as $x \to 0$ because the strategic bias dominates, while c^* gradually approaches the limiting value of $\frac{1}{3}$ when $x \to \infty$ (see Fig. 3(a)).

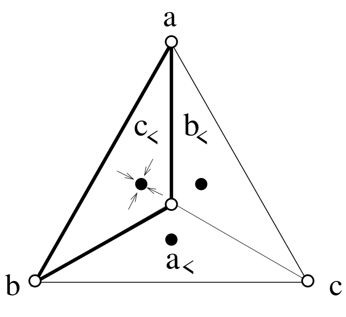

FIG. 2: Composition triangle showing the stable (dots) and unstable (circles) fixed points of the rate equations. The triangle represents the locus of points $a+b+c=1$ in the abc density space. Each sector is demarcated by separatrices, and the local flow near one stable fixed point is shown. The heavy lines outline the $c_<$ sector where the density of c is in the minority.

For the majority density, we write the first of Eqs. (2) as $\dot{a} + af = T$, where $f(t) \equiv 3T - rc(t)$, with formal solution

$$a(t) = T\,e^{-F(t)} \int_0^t e^{F(t')}\,dt' + a(0)\,e^{-F(t)}\ , \tag{5}$$

where

$$F(t) = \int_0^t f(t')\,dt = 3Tt - r\int_0^t \frac{c_- - c_+\mathcal{C}\,e^{-rt'/\tau}}{1-\mathcal{C}\,e^{-rt'/\tau}}\,dt' = -\ln\left[\frac{1-\mathcal{C}}{1-\mathcal{C}\,e^{-rt/\tau}}\right] + \frac{rt}{2\tau}[1-\tau(3x-1)].$$

Finally, we substitute the above result for $F(t)$ into Eq. (5) and perform the integral to obtain

$$a(t) = a^* - \frac{e^{-rt/\tau}}{1+\eta}\left[\frac{\mathcal{C}(1-\mathcal{C})}{1-\mathcal{C}e^{-rt/\tau}}\right] + \left[a(0) - \frac{2x\tau\,\{1+\mathcal{C}+\eta(1-\mathcal{C})\}}{(1-\mathcal{C})\,\{1-\eta^2\}}\right]\left[\frac{(1-\mathcal{C})e^{-rt(1-\eta)/2\tau}}{1-\mathcal{C}e^{-rt/\tau}}\right], \tag{6}$$

and similarly for $b(t)$. Here $\eta \equiv (1-3x)\tau$, and the steady-state densities are $a^* = b^* = (1-c^*)/2$. It is easy to check that $0 < 1-\eta < 2$, so that the approach to the steady-state is dominated by $a(t) - a^* \sim e^{-rt(1-\eta)/2\tau}$ in Eq. (6), while the minority species relaxes more quickly to stationarity: $c(t) - c^* \sim e^{-rt/\tau}$. The same considerations apply, *mutatis mutandis*, in the $a_<$ and $b_<$ sectors of the composition triangle.

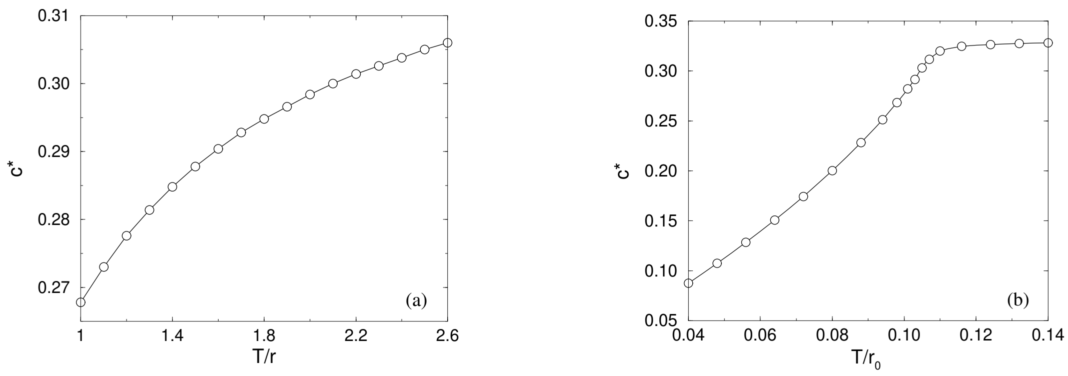

FIG. 3: Simulation results for the fixed point minority concentration c^* for: (a) a population of $N = 5000$ agents with constant strategic bias as a function of $x = \frac{T}{r}$, and (b) $N = 50000$ agents with state-dependent strategic bias as a function of $x_0 = \frac{T}{r_0}$. The numerical value c^*, which agrees perfectly with the rate equation predictions, is determined by the time integral of the concentration divided by the observation time. The change in behavior when $x_0 \approx \frac{1}{9}$ becomes a cusp as $N \to \infty$.

The non-trivial fixed points (dots in Fig. 2) are the basins of attractions for the $a_<$, $b_<$ and $c_<$ sectors for *any* value of the basic parameter x, and for almost all initial conditions (except for the special cases where all the densities initially equal or only two densities are nonzero).

B. State-Dependent Strategic Bias

We now consider strategic bias with state-dependent rate $r = r_0[(a+b)/2 - c]$ in the rate equations (2). To simplify technical details, we assume $a = b$ (corresponding to an initial state $a(0) = b(0)$) throughout the evolution; thus the normalization condition now is $2a + c = 1$. The rate equation for the minority density c in the $c_<$ sector now is:

$$\dot{c} = (1-3c)T - \frac{r_0}{2}c(1-c)(1-3c) \equiv -\frac{3r_0}{2}(c-c_-)(c-c_+)(c-c_3), \tag{7}$$

where $c_3 = \frac{1}{3}$, $c_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-8x_0}\right)$, and $x_0 \equiv \frac{T}{r_0}$. By plotting the right-hand side of Eq. (7) as a function of c, it is clear that the stable fixed point value is $c^* = c_-$ for $x_0 < \frac{1}{9}$. In this case, any initial state in the $c_<$ sector flows to the fixed point $(\frac{1-c^*}{2}, \frac{1-c^*}{2}, c^*)$; this behavior is qualitatively similar to that of constant strategic bias. Conversely, for $x_0 > \frac{1}{9}$ the stable fixed point is $c^* = \frac{1}{3}$, with c_+ and c_- both greater than $\frac{1}{3}$ (Fig. 3, right). Thus a sufficiently weak strategic bias or a sufficiently strong voting uncertainty, as quantified by $x_0 > \frac{1}{9}$, leads to the fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ becoming stable. This change in the stability of the fixed points as a function of the basic parameter x_0 is an unexpected feature of state-dependent strategic voting.

To solve for the time dependence, we perform a partial fraction expansion to transform the rate equation (7) to

$$dc\left[\frac{1}{c_3-c_-}\left(\frac{1}{c-c_3} - \frac{1}{c-c_-}\right) - \frac{1}{c_3-c_+}\left(\frac{1}{c-c_3} - \frac{1}{c-c_+}\right)\right] = \frac{3}{2}(c_+-c_-)r_0\,dt.$$

This equation can be straightforwardly integrated and the result is

$$\left(\frac{c(t)-c_3}{c(0)-c_3}\right)^{-\alpha_3}\left(\frac{c(t)-c_+}{c(0)-c_+}\right)^{-\alpha_+}\left(\frac{c(t)-c_-}{c(0)-c_-}\right)^{-\alpha_-}=e^{3(c_+-c_-)r_0t/2}\ , \tag{8}$$

where

$$\alpha_\pm=\frac{1}{c_3-c_\pm}\ , \qquad \alpha_3=\alpha_--\alpha_+=\frac{c_+-c_-}{(c_3-c_+)(c_3-c_-)}\ .$$

The main feature of this result is that the approach to the reactive steady state $c(t)\to c_-$ is still exponential in time, but the decay time can become quite long when the parameter $x_0\approx\frac{1}{9}$. For example for $x_0\lesssim\frac{1}{9}$, the asymptotic decay of the minority density is

$$c(t)-c_-\sim\exp\left[-\frac{3r_0}{2}(c_+-c_-)(c_--c_3)\,t\right]. \tag{9}$$

Thus as $x\to\frac{1}{9}$ where c_- and c_3 approach each other, the approach to the fixed point becomes very slow. On the other hand, for $x\gtrsim\frac{1}{9}$, the densities all decay to their common value of $\frac{1}{3}$ as

$$c(t)-c_3\sim\exp\left[-\frac{3r_0}{2}(c_3-c_+)(c_3-c_-)\,t\right]. \tag{10}$$

IV. INFLUENCE OF STOCHASTIC FLUCTUATIONS

In a finite population, the stochasticity of the dynamics allows the system to eventually escape the basin of attraction of a fixed point. This escape corresponds to a minority party becoming one of the top two parties on a slow time scale. To understand this escape dynamics, we formulate a discrete version of the strategic voter model for a fixed population N, with N_A agents of species A, N_B of species B, N_C of species C, with $N=N_A+N_B+N_C$. This population evolves by the reactions:

$$A\xrightarrow{T}\begin{cases}B\\C\end{cases},\quad B\xrightarrow{T}\begin{cases}A\\C\end{cases},\quad C\xrightarrow{T}\begin{cases}A\\B\end{cases},$$

$$AB\xrightarrow{r}\begin{cases}AA\\BB\end{cases},\qquad AC\xrightarrow{r}AA,\qquad BC\xrightarrow{r}BB,$$

for the case where C is in the minority. Corresponding rules apply when A and B are in the minority.

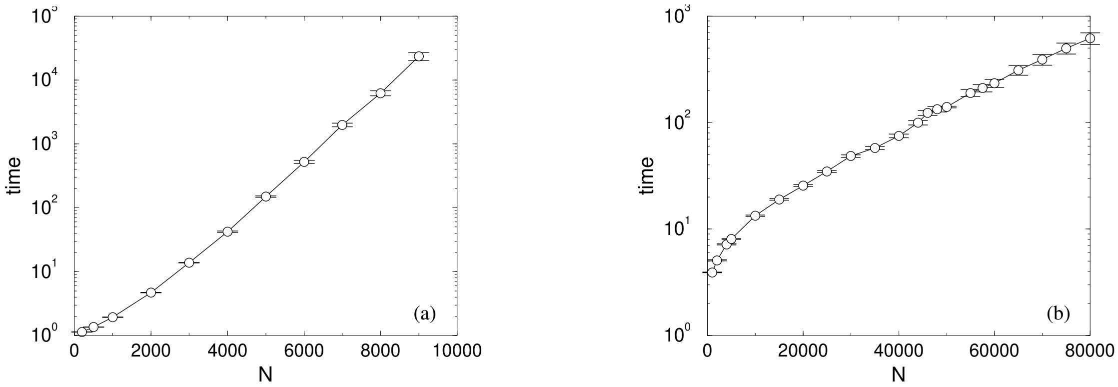

FIG. 4: Average time span for a single party to have minority status versus population N for strategic voting with: (a) constant strategic bias with $x=2$ and (b) state-dependent strategic bias with $x_0=0.104$; this latter value is chosen to give the same steady-state minority densities for both models. Both time dependences are consistent with exponential growth in N.

The stochastic dynamics of the strategic voter model may be analytically described by a master equation whose finite-size expansion [12, 13] leads to a Fokker-Planck equation. The fixed points in the rate equation then correspond, for finite N, to an effective attractor for the probability distribution. Because of finite-size fluctuations, the system can escape such a potential well and thereby move between different attractors. This barrier crossing corresponds to a change in the identity of the minority party. We can investigate this barrier crossing by rephrasing it as a first-passage problem [12, 13, 15]. Following a standard calculation (as given, for example, in Ref. [14]), we find that changes in the identity of the minority party occur on a time scale that grows exponentially in N [16], much longer than the time scale of fluctuations between the majority and second-place parties.

We have tested this prediction by numerical simulations. The time span over which a given state is in the minority as a function of the total population N does appear to grow exponentially with N for both constant and state-dependent strategic bias, but with a much smaller amplitude for the latter case (Fig. 4). For the more realistic case of a state-dependent strategic bias, simulations of single realizations exhibit changes in minority status on a time scale that can be tuned to be in a similar range as that of election data (Fig. 1).

V. DISCUSSION

We proposed a strategic voter model to describe the feature whereby two major parties dominate in certain parliamentary democracies, while a third party remains in the minority. Our model is based on a strategic bias that inherently disfavors the minority species. We studied two variants of strategic voting in which the strategic bias is either fixed or state dependent. For fixed bias, the minority species is doomed to remain eternally in the minority in the rate equation limit. For a state-dependent bias, a bifurcation occurs between a stable minority and equal densities of the three parties as the strength of the strategic bias varies.

It is worth mentioning that our three-choice voting model has an important distinction with predator-prey models that exhibit "competitive exclusion". This feature — known as Gause's Law [17] — states that two species that compete for the same resources cannot stably coexist if other ecological factors are constant. In our model, if we consider the opinions as different species, the dynamics allow for spontaneous mutations and additionally the interactions are not constant but state dependent. These features allow for a stable coexistence of all species.

For finite populations, stochastic fluctuations allow the minority party to escape its status on a time scale that grows exponentially with population size. The concomitant slow oscillations in the identity of the minority party can be readily seen in simulations, and this time evolution seems to roughly mirror real election data, especially if the latter could be extended over a much longer time scale. However, to quantitatively match the election data, it would be necessary to have a strategic bias that is still weaker than our state-dependent strategic voter model to reproduce both the density difference between majority and minority and the time scale for a change in minority status.

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