Voter Model on Heterogeneous Graphs

V. Sood[1,*] and S. Redner[1,†]

[1] *Theory Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

We study the voter dynamics model on heterogeneous graphs. We exploit the non-conservation of the magnetization to characterize how consensus is reached. For a network of N nodes with an arbitrary but uncorrelated degree distribution, the mean time to reach consensus T_N scales as $N\mu_1^2/\mu_2$, where μ_k is the $k^{\rm th}$ moment of the degree distribution. For a power-law degree distribution $n_k \sim k^{-\nu}$, T_N thus scales as N for $\nu > 3$, as $N/\ln N$ for $\nu = 3$, as $N^{(2\nu-4)/(\nu-1)}$ for $2 < \nu < 3$, as $(\ln N)^2$ for $\nu = 2$, and as $\mathcal{O}(1)$ for $\nu < 2$. These results agree with simulation data for networks with both uncorrelated and correlated node degrees.

PACS numbers: 02.50.-r, 05.40.-a, 89.75.Fb

In this letter we study the voter model [1] on heterogeneous networks and show that its behavior is dramatically different than that on regular lattices. Many recent studies of basic statistical mechanical models on heterogeneous graphs have begun to elucidate how the dispersity in node degree (the number of links attached to a node) affects critical behavior. A representative but incomplete set of examples include percolation [2], the Ising model [3–7], diffusion and random walks [8–11], as well as the voter model itself [12–15].

The voter model is perhaps the simplest and most completely solved example of cooperative behavior. For these reasons, our analytical predictions for the voter model on heterogeneous networks should provide new insights into the role of underlying heterogeneity on dynamical cooperative behavior. In the model, each node of a graph is endowed with two states – spin up and spin down. The evolution consists of: (i) picking a random voter; (ii) the selected voter adopts the state of a randomly-chosen neighbor. These steps are repeated until a finite system necessarily reaches consensus.

One basic property of the voter model is the exit probability, namely, the probability that the system ends with all spins up, $E_+(\rho_0)$, as a function of the initial density of up spins ρ_0. Because the mean magnetization (averaged over all realizations and all histories) is conserved on regular lattices, and because the only possible final states are consensus, $E_+(\rho_0) = \rho_0$ [1]. A second basic property is the mean time to reach consensus, T_N. For regular lattices in d dimensions, it is known that T_N scales with the number of nodes N as N^2 in $d = 1$, as $N \ln N$ in $d = 2$, and as N in $d > 2$ [1, 16]. For heterogeneous networks, we find that T_N grows as $N\mu_1^2/\mu_2$, where μ_k is the $k^{\rm th}$ moment of the degree distribution of the network [Eq. (13)]. In contrast to lattice systems, the N dependence of T_N is generally sublinear.

To understand how dispersity in node degree affects voter model dynamics, we first examine the illustrative example of the complete bipartite graph. We then extend this approach to determine the behavior of the voter model on networks with power-law degree distributions, but with no correlations between node degrees. Finally, we validate our theoretical results by simulations of the voter model on networks with power-law degree distributions, both with and without node degree correlations.

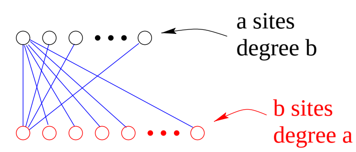

FIG. 1: The complete bipartite graph $K_{a,b}$.

Consider the voter model on the complete bipartite graph $K_{a,b}$ of $N = a + b$ nodes that are partitioned into two subgraphs a and b (Fig. 1). Each node in the a subgraph is connected to all nodes in the b subgraph, and vice versa. Let $N_{a,b}$ be the respective number of up spins on each subgraph. In an update event, these numbers change according to

$$\begin{aligned} dN_a &= \frac{a}{a+b}\Big[\frac{a-N_a}{a}\frac{N_b}{b} - \frac{N_a}{a}\frac{b-N_b}{b}\Big], \\ dN_b &= \frac{b}{a+b}\Big[\frac{b-N_b}{b}\frac{N_a}{a} - \frac{N_b}{b}\frac{a-N_a}{a}\Big]. \end{aligned} \qquad (1)$$

For dN_a, the gain term accounts for flipping a down spin in subgraph a due to its interaction with an up spin in b, while the loss term accounts for flipping an up spin in subgraph a. The second equation accounts for the evolution of N_b. Since the time increment for an event is proportional to $1/(a + b)$, the subgraph densities $\rho_a = N_a/a$ and $\rho_b = N_b/b$ obey $\dot\rho_{a,b} = \rho_{b,a} - \rho_{a,b}$, with solution

$$\rho_{a,b}(t) = \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\, e^{-2t} + \frac{1}{2}[\rho_a(0) + \rho_b(0)]. \qquad (2)$$

While the sum of the subgraph densities, $\rho_a + \rho_b$ is conserved, the magnetization $m = (a\rho_a + b\rho_b)/(a + b)$ is not [14]. However, the bias in the rate equations for ρ_a and ρ_b drive the subgraph densities to the common value $\rho_{a,b}(\infty) = \frac{1}{2}[\rho_a(0) + \rho_b(0)]$ and magnetization conservation is restored as this final state is approached. It also bears mentioning that the magnetization itself is conserved if the update rule is link-based [14].

Since $\rho_a + \rho_b$ is conserved, the sum of the subgraph densities in the final state equals 2 with probability E_+. Thus the exit probability is

$$E_+ = \frac{1}{2}[\rho_a(0) + \rho_b(0)]. \tag{3}$$

When the initial spins on the two subgraphs are oppositely oriented, there is an equal probability of ending with all spins up or all spins down, *independent* of the subgraph sizes. In the extreme case of the star graph $K_{a,1}$, with $a \gg 1$ up spins at the periphery and a single down spin at the center, there is only a 50% change that the system ends with all spins up.

We now study the mean time until consensus $T_N(\rho_a, \rho_b)$ – either all spins up or all spins down – as a function of N, ρ_a, and ρ_b. This consensus time satisfies the recursion formula [17, 18]:

$$\begin{aligned} T_N(\rho_a,\rho_b) \;=\; & \mathbf{P}(\rho_a,\rho_b \to \rho_a \pm \frac{1}{a},\rho_b)[T_N(\rho_a \pm \frac{1}{a},\rho_b) + \delta t] \\ +\; & \mathbf{P}(\rho_a,\rho_b \to \rho_a,\rho_b \pm \frac{1}{b})[T_N(\rho_a,\rho_b \pm \frac{1}{b}) + \delta t] \\ +\; & \mathbf{P}(\rho_a,\rho_b \to \rho_a,\rho_b),[T_N(\rho_a,\rho_b) + \delta t], \end{aligned} \tag{4}$$

where $\delta t = 1/(a+b) \equiv 1/N$ is the time step for a single spin-flip attempt. For example, the first term (a shorthand for two contributions) accounts for flipping a down (up) spin in subgraph a so that $\rho_a \to \rho_a \pm \frac{1}{a}$. The probability for flipping a down spin in subgraph a is $\mathbf{P}(\rho_a, \rho_b \to \rho_a + \frac{1}{a}, \rho_b) = \frac{a}{a+b}(1-\rho_a)\rho_b$, where $\frac{a}{a+b}(1-\rho_a)$ is the probability to choose a down spin in subgraph a and ρ_b is the probability to choose an up spin in subgraph b. This equation is subject to the boundary conditions $T_N(0,0) = T_N(1,1) = 0$.

Expanding this recursion formula to second order, we find, after straightforward algebra,

$$\begin{aligned} N\delta t \;=\; & (\rho_a - \rho_b)(\partial_a - \partial_b)T_N(\rho_a,\rho_b) \\ & -\frac{1}{2}(\rho_a + \rho_b - 2\rho_a\rho_b)\left(\frac{1}{a}\partial_a^2 + \frac{1}{b}\partial_b^2\right)T_N(\rho_a,\rho_b) \end{aligned} \tag{5}$$

where ∂_i denotes a partial derivative with respect to ρ_i. The first term on the right accounts for a convection that drives the system to equal subgraph magnetizations in a time of order one. Subsequently, diffusive fluctuations govern the ultimate approach to consensus (see Fig. 2). We thus compute the consensus time by replacing the subgraph densities ρ_a and ρ_b by their common value ρ. In so doing, we ignore the initial transient for $t \sim O(1)$, during which the subgraph densities are unequal. We also transform the derivatives with respect to ρ_a and ρ_b in Eq. (5) to derivative with respect to ρ to yield

$$\frac{1}{4}\rho(1-\rho)\left(\frac{1}{a} + \frac{1}{b}\right)\partial^2 T_N = -1, \tag{6}$$

with solution

$$T_N(\rho) = -\frac{4ab}{a+b}\left[(1-\rho)\ln(1-\rho) + \rho\ln\rho\right] \tag{7}$$

Notice that if $a = \mathcal{O}(1)$ and $b = \mathcal{O}(N)$ (star graph), the consensus time $T_N \sim \mathcal{O}(1)$, while if both a and b are $\mathcal{O}(N)$, then $T_N \sim \mathcal{O}(N)$, as on a complete graph.

We now extend this approach to graphs with arbitrary degree distributions but without degree correlations, *i.e.*, we treat all nodes with the same degree as equivalent [19]. We define ρ_k as the density of up spins in the subset of nodes of degree k. Similar to Eq. (4), the recursion for the mean consensus time on a heterogeneous graph, with initial densities $\{\rho_k\}$, is:

$$\begin{aligned} T_N(\{\rho_k\}) \;=\; & \sum_k \mathbf{P}(k; \mp \to \pm)[T_N(\rho_k \pm \delta_k) + \delta t] \\ +\; & \sum_k \mathbf{Q}(\{\rho_k\})[T_N(\{\rho_k\}) + \delta t], \end{aligned} \tag{8}$$

where $\mathbf{P}(k; \mp \to \pm)$ is the probability that a spin down (up) on a node of degree k flips in an update, $\mathbf{Q}$ is the probability of no flip, and $\delta_k = 1/(Nn_k)$ is the change in ρ_k when a spin flip occurs at a site of degree k. Here n_k is the fraction of sites with degree k.

Since the probability of choosing a node is $1/N$, the spin flip probability may be written as

$$\mathbf{P}(k; \mp \to \pm) \;=\; \sum_{x:\left(\substack{k_x=k \\ s_x=\mp}\right)} \frac{1}{N} \sum_{y:s_y=\pm} \frac{1}{k}\mathrm{A}_{xy}\,, \tag{9}$$

where A_{xy} is the adjacency matrix element between nodes x and y ($A_{xy} = 1$ if x and y are connected and $A_{xy} = 0$ otherwise). Thus the second sum is the probability that a node x with degree k chooses a neighbor with spin up (down). Under the mean-field assumption that neighboring node degrees are uncorrelated, we write A_{xy} as $k_x k_y/\mu_1 N$, where $\mu_1 \equiv \sum_k k n_k$ is the mean node degree for the graph. That is, we replace A_{xy} by the probability that an edge between node x of degree k_x and node y of degree k_y exists. Then the second sum in Eq. (9) for spin up and spin down simplifies respectively to

$$\begin{aligned} \frac{1}{\mu_1 N}\sum_{\substack{y \\ s_y=+}} k_y \;&=\; \frac{1}{\mu_1}\sum_j j n_j \rho_j \equiv \omega \\ \frac{1}{\mu_1 N}\sum_{\substack{y \\ s_y=-}} k_y \;&=\; \frac{1}{\mu_1}\sum_j j n_j(1-\rho_j) \equiv 1-\omega. \end{aligned}$$

Namely, we decompose the nodes y according to their degree, and we define ω as the average degree-weighted density of up spins. In this formulation, each spin of given sign flips with the same probability that is a function of the degree-weighted magnetization rather than of the global magnetization, as in the case for degree-regular graphs. Since the first sum in Eq. (9) gives the density of down (up) spin in the subset of nodes with degree k, we now write $\mathbf{P}(k; - \to +) = n_k \omega (1-\rho_k)$, and similarly, $\mathbf{P}(k; + \to -) = n_k (1-\omega)\rho_k$. Finally, the probability that there is no change in a single spin flip attempt is $\mathbf{Q}(\{\rho\}) = 1 - \sum_k (\mathbf{P}(k; - \to +) + \mathbf{P}(k; + \to -))$.

These simplifications enable us to write Eq. (8) as

$$\begin{aligned} -\delta t \, &= \sum_k n_k \left[\omega(1-\rho_k)(T(\rho_k+\delta_k) - T(\{\rho_k\}))\right] \\ &+ \sum_k n_k \left[(1-\omega)\rho_k(T(\rho_k-\delta_k) - T(\{\rho_k\}))\right] \end{aligned} \quad (10)$$

Expanding this recursion to second order we obtain

$$N\delta t = \sum_k (\rho_k - \omega)\partial_k T - \sum_k \frac{(\omega + \rho_k - 2\omega\rho_k)}{2Nn_k}\, \partial_k^2 T, \quad (11)$$

where ∂_k denotes the partial derivative with respect to ρ_k. The convective terms on the right-hand side again drive the system to the state where ρ_k is equal to the weighted magnetization ω for all k.

To check this convective behavior, we followed the evolution of single realizations of the voter model on scale-free graphs with degree distribution $n_k \propto k^{-2.5}$ and mean degree $\mu_1 = 8$ generated according to the Molloy-Reed (MR) model [20]. Each node is assigned a random number of stubs k that is drawn from a specified degree distribution. Pairs of unlinked stubs are then randomly joined. This construction eliminates degree correlations between neighboring nodes. For the initial state, we assign all nodes with degree larger than μ_1 as spin down and all remaining nodes as spin up. A plot of the spin up densities ρ_6 and ρ_{11} for nodes of degrees 6 and 11 versus the degree-weighted up-spin density shows that these "subgraph" densities quickly approach equal values (Fig. 2). Analogous behavior occurs on the bipartite graph and on scale-free networks with degree correlations.

For long times, we thus drop the convective terms and set $\rho_k = \omega \ \ \forall k$. Concomitantly, we transform the partial derivatives with respect to k to derivatives with respect to ω by using $\partial_k \omega = n_k k/\mu_1$ to reduce (11) to

$$\frac{1}{N}\sum_k \left(\frac{k^2}{\mu_1^2} n_k\right) \omega(1-\omega)\ \partial_\omega^2 T = -1. \quad (12)$$

Since $\sum_k k^2 n_k = \mu_2$, the second moment of the degree distribution, this equation can be reduced to a similar form to (6), with solution

$$T_N(\omega) = -N\frac{\mu_1^2}{\mu_2}\,\left[(1-\omega)\ln(1-\omega) + \omega\ln\omega\,\right]. \quad (13)$$

For a scale-free network [21] with degree distribution $n_k \sim k^{-\nu}$, the m^{th} moment is $\mu_m \sim \int^{k_{\max}} k^m n_k \, dk$. Here

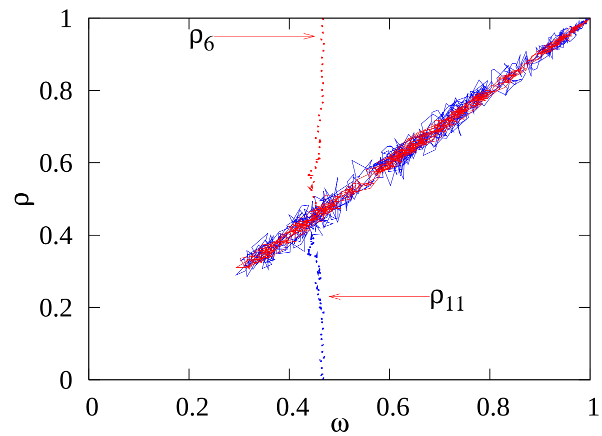

FIG. 2: Trajectories of $\rho_6(t)$ (degree less than μ_1) and $\rho_{11}(t)$ (degree greater than $\mu_1 = 8$) versus ω, for one realization of the voter model on a network of 2×10^5 nodes, with degree distribution $n_k \sim k^{-2.5}$. The initial state is $(\rho_{k>\mu_1}(0), \rho_{k\leq\mu_1}(0)) = (0,1)$. The dotted curves are the initial transient for $t \lesssim 1$, after which diffusive motion leads to consensus at $(1,1)$.

$k_{\max} \sim N^{1/(\nu-1)}$ is the maximal degree in a finite network of N nodes; this is obtained from the extremal condition $\int_{k_{\max}} k^{-\nu}\, dk = N^{-1}$ [22]. Thus the second moment diverges at the upper limit for $\nu \leq 3$ while the first moment diverges for $\nu \leq 2$.

Assembling the results for the moments, the mean consensus time on a scale-free graph has the N dependence

$$T_N \sim \begin{cases} N & \nu > 3, \\ N/\ln N & \nu = 3, \\ N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\ (\ln N)^2 & \nu = 2, \\ \mathcal{O}(1) & \nu < 2. \end{cases} \quad (14)$$

The prediction $T_N \sim N/\ln N$ for $\nu = 3$ may explain the apparent power law $T_N \sim N^{0.88}$ reported in a previous simulation of the voter model on such a network [14].

To test our predictions, we simulated the voter model on the Molloy-Reed (MR) network [20] and on the growing network with redirection (GNR) [23]. The GNR is built by adding nodes sequentially, where each new node attaches either to a randomly-selected node with probability $1-r$ or to the ancestor of this target with probability r. We chose the out degree of each node to be 4, and redirection was applied to each outgoing link of the new node. This construction gives a network with a power-law degree distribution $n_k \propto k^{-\nu}$, with $\nu = 1 + \frac{1}{r}$ in the range $(2, \infty)$ as r is varied between 0 and 1.

Fig. 3 shows the N dependence of T_N for representative values of the degree exponent ν for both the MR network and the GNR. The results for the two networks with the same ν are extremely close, suggesting that degree correlations have a small effect on voter model dy-