A Guide to First-Passage Processes

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- 1.4 Relation Between Laplace Transforms and Real Time Quantities
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 - 1.6.2 Green's Function Formalism Hitting Probability
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 - Absorption Mode
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Direct Solution

Recursive Solution

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