# Superballistic motion in a "random-walk" shear flow

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We investigate the motion of a random walker that is driven by a "random-walk" shear flow. This unidirectional two-dimensional flow is defined by a velocity field in the $x$ direction, which depends only on the transverse position $y$, and whose magnitude $v=v_x(y)$ is given by the displacement of a random walk of $y$ steps. For this model, the root-mean-square longitudinal displacement of a diffusing particle that is passively carried by the flow increases as $t^{5/4}$. In a single configuration of the random shear, the probability distribution of displacements is bimodal, while the distribution function averaged over many configurations has a single cusped peak at the origin. As a consequence, the configuration-averaged probability that a walk is at $x=0$ decays more slowly than the $t^{-5/4}$ dependence that would be expected on the basis of single-parameter scaling. The large-distance decay of the average probability distribution is also found to be anomalously slow. These unusual features can be explained on the basis of a scaling argument together with an effective-medium-type approximation. Our results are confirmed by numerical simulations.

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## I. INTRODUCTION

Considerable effort has been devoted to understanding the motion of random walks in disordered media. In many cases, this motion is subdiffusive, i.e., the mean-square displacement of a random walk, $\langle r^2(t)\rangle$, grows more slowly than linearly with time $t$. This basic phenomenon arises because the hopping rates are effectively length-scale dependent, and they vanish as a power of the length scale. This fundamental aspect has stimulated investigations that have elucidated many of the intriguing features of anomalously slow diffusion phenomena [1–4].

The complementary situation of superdiffusive stochastic motion, where the mean-square displacement grows faster than linearly with time, has received less attention. However, there are a variety of situations where this type of phenomenon should occur. One example is the dispersion of dynamically neutral tracer particles about the mean flow in a nonuniform convection field. To describe this dispersion, Matheron and de Marsily [5] introduced a simple model of a stratified two-dimensional flow, where the longitudinal velocity (along $x$) is a random white-noise function only of the transverse coordinate $y$. In a reference frame at rest with respect to the tracer, this flow field leads to an effective diffusivity which is an increasing function of the length scale. Consequently, the root-mean-square (rms) longitudinal displacement of a diffusing particle in this flow, $x_{\rm rms}$, grows faster than diffusively [5,6], $x_{\rm rms}\sim t^{\nu}$, with the size exponent $\nu=\frac{3}{4}$. Furthermore, the configuration-averaged probability distribution of longitudinal displacements decays at large distances as [7,8] $\langle P(x,t)\rangle \sim \exp[-(x/t^{\nu})^{\delta}]$, with $\delta=\frac{4}{3}$. This exponent is much smaller than the value that is expected on the basis of the general relation $\delta=(1-\nu)^{-1}$ characteristic of a broad class of stochastic transport processes [9,10]. In the case of the Matheron–de Marsily model, the value $\nu=\frac{3}{4}$ would imply $\delta=4$; consequently the large-distance decay of the probability distribution in this model is anomalously slow.

These unusual features motivate our investigation of a transport model in the spirit of Matheron and de Marsily

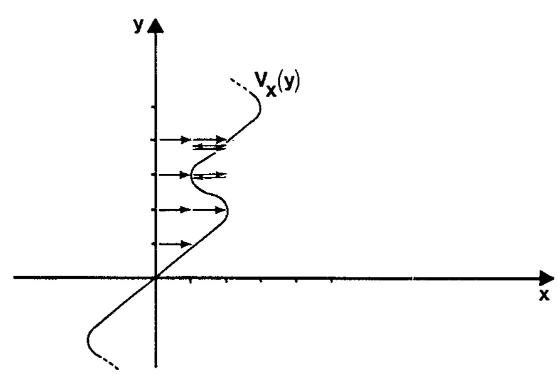


FIG. 1. A typical realization of the velocity field in random shear flow.

 

which gives rise to *superballistic* motion, where $x_{\rm rms}$ grows faster than linearly with time. Our model involves a random walk moving in a steady two-dimensional "random-walk" shear flow, in which the magnitude of the velocity field is itself determined by a random walk. We define the longitudinal velocity (along $x$) to depend only on the transverse coordinate $y$, and to equal the displacement of a random walk of $y$ steps (Fig. 1). For reasons of convenience, we take the convection field to be antisymmetric about $y=0$, i.e., $v_x(-y)=-v_x(y)$. Typically, then, the longitudinal velocity increases with transverse length scale as $y^{1/2}$, and the increasing velocity as a function of the length scale leads to superballistic motion. More interestingly, the probability distribution of displacements is bimodal for a single configuration of the random shear flow, but the configuration-averaged distribution is unimodal, with a logarithmic singularity at the origin. These results can be explained, to a large degree, by a scaling argument applied to an effective-medium-type approximation for the flow field. In Sec. II we shall present some basic phenomenology of our model and in Sec. III we give our theoretical arguments for the form of the longitudinal probability distribution function.

## II. BASIC RESULTS

To estimate the time dependence of the longitudinal rms displacement, note that at time $t$, a typical random walk has diffused a distance $y \sim t^{1/2}$ in the transverse direction. At this transverse position, a typical shear configuration has a longitudinal bias $v_x(y(t)) \sim y(t)^{1/2} \sim t^{1/4}$. Consequently, we expect $x_{\rm rms}$ to grow as

$$x_{\rm rms} \approx v(t)t \propto t^{5/4} . \quad (1)$$

At a more rigorous level, one can straightforwardly apply the path-integral formalism of Refs. [7,8] to verify the exponent in Eq. (1), and also to show that the $k$th moment of the longitudinal displacement scales as $x_{\rm rms}^k$.

These expectations are confirmed both by exact enumeration numerical studies and by Monte Carlo simulations of many walks on typical shear-flow configurations (Fig. 2). In the exact enumeration, we considered a system of width $w=42$ with free boundary conditions in the transverse direction, and with an antisymmetric random shear-flow field about the middle of the strip, $y=0$. For a system of this width there are $2^{(w-2)/2}=2^{20}$ independent configurations of the flow. The exact random-walk probability distribution walk is propagated forward in time for each configuration, with the initial condition of a single random walk starting at $y=0$, and then an average over all shear configurations is performed. The enumeration can be regarded as being exact for an infinite system, as long as the transverse displacement remains within the strip, i.e., for up to 22 time steps. Our Monte Carlo simulations involved following the motion of many random walks of up to 1000 time steps, and then averaging over many configurations of the flow field. Both of these numerical approaches indicate that $\langle x(t)^{2k}\rangle^{1/2k} \propto x_{\rm rms}$. In the exact enumeration, the slopes of successive pairs of data points for $\langle x(t)^{2k}\rangle^{1/k}$, when plotted on a double logarithmic scale, are initially

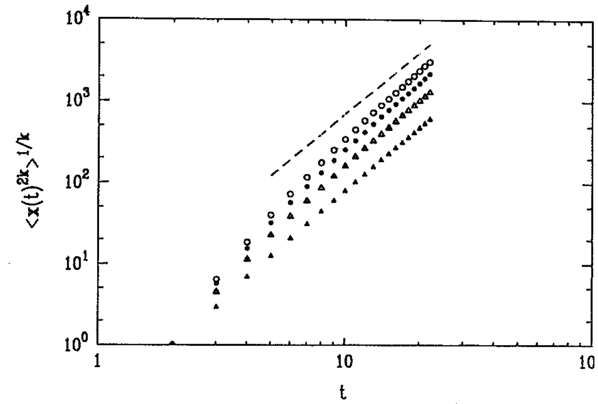


FIG. 2. Double logarithmic plot of $\langle x(t)^{2k}\rangle$ vs time in random shear flow. The data are based on exact enumeration over all walks of up to 22 steps in an exact average over all flow configurations in a system of width $w=42$. Shown are the cases $k=1$ (▲), $k=2$ (△), $k=3$ (●), and $k=4$ (○). The dashed line has slope $\frac{5}{2}$.

greater than the expected value of $\frac{5}{2}$ but slowly approach $\frac{5}{2}$ as $t$ increases.

More interesting behavior is exhibited by the probability distribution of longitudinal displacements, summed over the transverse coordinate [Fig. 3(a)]. Since $x_{\rm rms}$

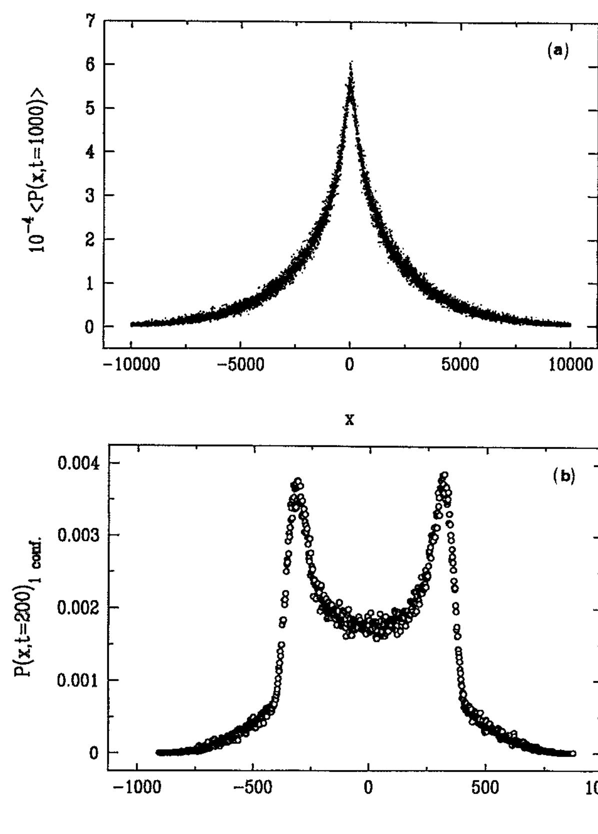


FIG. 3. The probability distribution of longitudinal displacements in random shear flow at $t=1000$ time steps, (a) averaged over 1000 walks per configuration and then over 1000 configurations of the random shear, and (b) averaged over 250 000 walks of 200 steps in a representative single configuration of random shear.

grows as $t^{5/4}$, it is natural to hypothesize that the configuration average of this probability distribution, $\langle P(x,t)\rangle$, has the scaling form

$$\langle P(x,t)\rangle \sim t^{-5/4}\Phi(x/t^{5/4}) , \tag{2}$$

with the scaling function possessing the large-distance cutoff,

$$\Phi(z)\to e^{-z^{\delta}}, \quad z\to\infty , \tag{3}$$

and with $\Phi(z)$ approaching a constant value as $z\to 0$.

Simulations of the longitudinal distribution reveal several interesting properties which are not expected on the basis of this naive scaling ansatz. First, there is a sharp cusp in the distribution at the origin. This cusp occurs when averaging over all possible environments for narrow systems, and also when averaging over a large number of typical environments for wider systems. Perhaps a more striking feature is the disparity between the unimodal form of $\langle P(x,t)\rangle$ and the bimodal form of the probability distribution in a single typical environment [Fig. 3(b)]. (This difference still occurs if the velocity field is not antisymmetric about $y=0$. In this general case, the distribution still has two peaks which are no longer symmetric about $x=0$.) This lack of self-averaging is characteristic of the probability distribution in many disordered systems [11]. In the Matheron–de Marsily model, for example, the width of the probability distribution in a typical environment and the width of the configuration-average probability distribution grow at the same rate, but with different prefactors [8]. However, in the random walk shear-flow model, even qualitative aspects of the shape of the distribution are not self-averaging.

The cusp in the small-distance behavior of the distribution function also manifests itself in the time dependence of $\langle P(x=0,t)\rangle$, the configuration-averaged probability that the longitudinal displacement equals zero at time $t$. If $\Phi(z)$ were nonsingular, then Eq. (2) predicts that $\langle P(0,t)\rangle$ will decay as $t^{-5/4}$. However, the numerical data indicate that the return probability decays at a slower rate (Fig. 4). We shall argue in the next section that this feature arises from a logarithmic singularity in $\Phi(z)$ as $z\to 0$.

The large-distance form of the probability distribution can be quantified by examining dimensionless ratios of high-order moments of the longitudinal displacement. For this purpose we consider

$$p_{2k}(t)\equiv\frac{\langle x^{2k}(t)\rangle}{\langle x^{2k-2}(t)\rangle^{k/(k-1)}} . \tag{4}$$

Since $\langle x^{2k}(t)\rangle \sim x_{\rm rms}^{2k}$, the $p_{2k}(t)$ are indeed dimensionless, and their limiting $t\to\infty$ numerical values are increasingly determined by the tail of the probability distribution as $k$ becomes large. Furthermore, the $p_{2k}(t)$ computed from enumeration have no statistical error, so that we can extrapolate to $t\to\infty$ with reasonable accuracy. Using rudimentary extrapolation methods, we thus estimate $p_4^\infty\approx 5.50$, $p_6^\infty\approx 5.00$, $p_8^\infty\approx 4.74$, $p_{10}^\infty\approx 4.65$, $p_{12}^\infty\approx 4.56$, and $p_{14}^\infty\approx 4.53$, with a subjective error estimate of $\pm 3$ in the last decimal place.

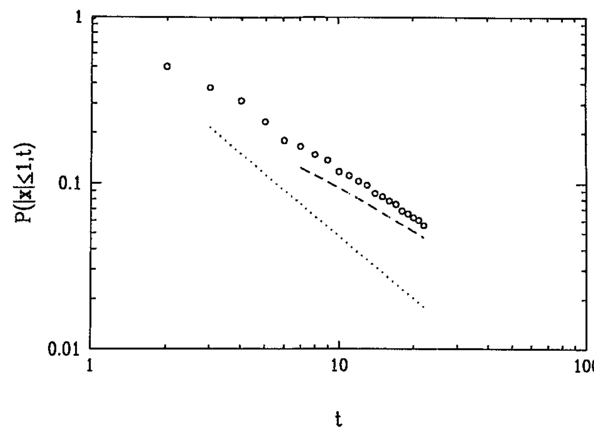


FIG. 4. Exact enumeration data for the probability that the longitudinal displacement equals zero at time $t$ on a double logarithmic scale. To reduce the even-odd fluctuations in this probability, the quantity $\langle P(|x|\le 1,t)\rangle$ is actually shown (○). The dotted line corresponds to $t^{-5/4}$, while the dashed line corresponds to $t^{-5/4}\ln t$.

On the other hand, computing the moments directly from Eq. (3) gives the following expression for the moment ratios:

$$p_{2k}^{\infty}=\frac{\Gamma((2k+1)/\delta)}{\Gamma((2k-1)/\delta)^{(k/k-1)}\Gamma(1/\delta)^{1/(k-1)}} . \tag{5}$$

This expression is appropriate in the scaling regime where $z\equiv x/t^{5/4}\sim O(1)$. Since the higher-order moments are determined by large values of the displacement, the scaling regime therefore also corresponds to $t\to\infty$, as denoted by the superscript in Eq. (5). By matching the numerical estimates for $p_{2k}^\infty$ for $2k=4,6,\ldots$ with those found from Eq. (5), we infer a $\delta$ value for each $k$. Our numerical estimates for $p_{2k}^\infty$ correspond, respectively, to the $\delta$ values 1.035, 1.100, 1.155, 1.185, 1.212, and 1.225.

Since the shape exponent inherently reflects the large-distance tail of the distribution, we expect that the estimate for $\delta$ obtained from $p_{2k}^\infty$ will become progressively more accurate as $k$ becomes large. A plot of the above-quoted $\delta$ values versus $1/k$ shows a trend toward approximately $\frac{4}{3}$ as $1/k\to 0$. We shall argue below that the shape exponent actually has the numerical value $\frac{4}{3}$, a result which arises from the contribution of walks in rare configurations of the random shear flow to the average probability distribution.

## III. SCALING BEHAVIOR OF $\langle P(x,t)\rangle$

### A. Effective-medium approximation

To account for both the value of the exponent $\delta$ and also the cusp in $\langle P(x=0,t)\rangle$, we develop an effective-medium type of approximation for the random shear flow by replacing the velocity field of a given (antisymmetric) environment with an idealized "split" flow, in which the velocity is $+v_{\rm eff}$ for $y>0$, and $-v_{\rm eff}$ for $y<0$ (Fig. 5). We then choose the value of $v_{\rm eff}$ to match the typical bias encountered by a random walk at time $t$. Thus for a random walk at transverse position $y(t)$, the typical value of

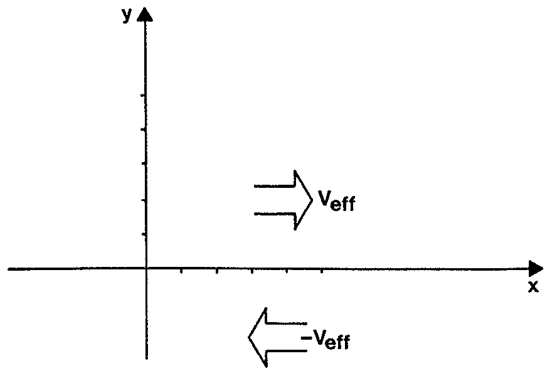


FIG. 5. The split medium approximation for the velocity field in random shear flow.

$v_x(y(t))$ in a typical environment is proportional to $y(t)^{1/2}$. Furthermore, since the distribution of velocities in the random shear flow at a fixed value of $y(t)$ is Gaussian, we also assume that the distribution of $v_{\rm eff}$ in the effective medium is Gaussian.

For the split flow velocity field, the probability distribution of longitudinal displacements, averaged over all transverse displacements, is identical to the classical arcsine law for the probability distribution of long leads in a one-dimensional random walk [12]. That is, the probability that a diffusing particle has a displacement equal to $x$ in split flow, $P_{\rm SF}$ is the same as the probability that the particle spends a time $(t+x)/2$ with $y>0$ and a time $(t-x)/2$ with $y<0$. Based on this equivalence, one has, from Ref. [12],

$$P_{\rm SF}(x,t)\sim\frac{1}{\pi v_{\rm eff}t[1-(x/v_{\rm eff}t)^2]^{1/2}} . \tag{6}$$

An amusing, and at first sight, unexpected, feature of this distribution is that the maximum occurs at the extremal displacement. This bimodal form also provides a reasonable qualitative description of the longitudinal probability distribution for a single typical configuration of the random shear flow.

In our effective-medium approach, we therefore take $P_{\rm SF}(x,t)$ as representing the longitudinal probability distribution of a single configuration of the shear flow, from which the configuration average is computed. Within our approximation, this computation involves averaging over the distribution of $v_{\rm eff}$ and the distribution of transverse displacements. This gives

$$\langle P(x,t)\rangle\approx\int_{-\infty}^{\infty}dy\frac{1}{\sqrt{t}}e^{-y^2/t}\int_{x/t}^{\infty}dv_{\rm eff}\frac{1}{\sqrt{|y|}}e^{-v_{\rm eff}^2/|y|}\frac{1}{\pi v_{\rm eff}t[1-(x/v_{\rm eff}t)^2]^{1/2}} . \tag{7}$$

This integral can be put into a convenient dimensionless form by introducing $u\equiv y/t^{1/2}$ and $w\equiv v_{\rm eff}t/x$ to yield

$$\langle P(x,t)\rangle\propto t^{-5/4}\int_0^{\infty}du\frac{1}{\sqrt{u}}e^{-u^2}\times\int_1^{\infty}dw\,e^{-z^2w^2/u}\frac{1}{(w^2-1)^{1/2}} , \tag{8}$$

where $z\equiv x/t^{5/4}$. Thus our effective-medium approximation immediately leads to the scaling form for $\langle P(x,t)\rangle$, hypothesized in Eq. (2).

The asymptotic behavior of $\langle P(x,t)\rangle$ can now be found directly from Eq. (8). For $z\to 0$, the second integral is cut off at $w\approx u^{1/2}/z$, and the dominant contribution to this integral arises for values of $w$ close to this upper limit. Accordingly, the integral can be approximated as

$$t^{-5/4}\int_0^{\infty}du\frac{1}{\sqrt{u}}e^{-u^2}\int_1^{u^{1/2}/z}\frac{dw}{w} , \tag{9}$$

which diverges logarithmically in $z$. Thus even though $P_{\rm SF}(x,t)$, which corresponds to the longitudinal probability distribution in a single configuration of shear flow, is bimodal, the average over all configurations within the effective-medium approximation leads to a single cusped peak at the origin.

In the complementary situation of $z\to\infty$, we can evaluate the integral by the Laplace method. The integral over $u$ gives rise to a controlling factor proportional to $\exp[-(zw)^{4/3}]$, and the integral over $w$ is asymptotic to the integrand evaluated at $w=1$. Consequently, we find the large-distance behavior of the scaling function to be

$$\langle P(x,t)\rangle\sim e^{-z^{4/3}},\quad z\to\infty , \tag{10}$$

from which we conclude that the shape exponent $\delta=\frac{4}{3}$.

The asymptotic behaviors of the probability distribution can also be obtained by an alternative effective-medium approximation which is in the same spirit as that outlined above. In this alternative approach, a given configuration of the random shear is replaced by a linear shear, with the magnitude of the velocity in this effective medium again chosen to match that encountered by the random walk in the actual environment at time $t$. The end result of this approach yields a distribution whose asymptotics coincide with Eqs. (9) and (10).

### B. Large-distance decay of $\langle P(x,t)\rangle$

For many random-walk processes [9,10], the size and shape exponents, $\nu$ and $\delta$, respectively, satisfy $\delta=(1-\nu)^{-1}$. In the domain of applicability of this result, the relation can be deduced simply from the observation that walks which are completely stretched contribute to the tail of the probability distribution. If the distri-

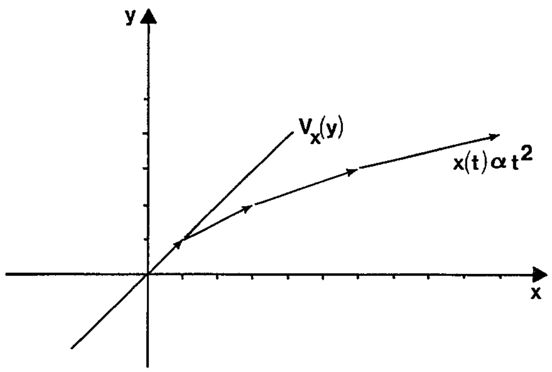


FIG. 6. The extreme environment, and the trajectory of the extreme walk that contributes to the large-distance tail of $\langle P(x,t)\rangle$.

bution of displacements has the scaling form $P(x,t)\sim\exp[-x/t^{\nu})^{\delta}]$, then the probability of finding a stretched walk is, therefore, $p(x\sim t,t)\sim\exp(-t^{(1-\nu)\delta})$. On the other hand, a completely stretched walk is constructed by choosing only one direction at each step of a walk, and clearly this leads to a probability at time $t$ which decays as $e^{-at}$. Equating these two forms yields $\delta=(1-\nu)^{-1}$. This relation fails in the case of the Matheron and de Marsily model, where the anisotropic nature of the problem plays a crucial role in determining the tail of the distribution function. Clearly, the relation $\delta=(1/\nu)^{-1}$ also fails in random shear flow, where $\nu>1$. In this latter case, we present a Lifshitz singularity argument, similar in spirit to that applied in the Matheron and de Marsily model, to determine the value of the shape exponent $\delta$.

For random shear, walks with the largest value of the longitudinal displacement arise in an extreme environment of purely linear shear, which is achieved by the underlying random-walk generator for the flow having a straight trajectory (Fig. 6). In this velocity field, the largest value of $x(t)$ clearly occurs when the diffusing particle takes transverse steps that are all in the same direction, i.e., $y(t)=t$. At time $t$, such a walk therefore has a longitudinal displacements $x(t)\sim\int^t v_x(y)dy\sim t^2$. The overall probability of constructing this extreme walk equals the probability of finding a pure shear region of width $t$, $(e^{-t})$ times the probability of having all transverse steps in the same direction (also $e^{-t}$). Consequently, $\langle P(x\sim t^2,t)\rangle\sim e^{-2t}$. On the other hand, if

$$\langle P(x,t)\rangle\sim\exp[-(x/t^{5/4})^{\delta}] ,$$

then

$$\langle P(x\sim t^2,t)\rangle\sim e^{-(t^{3/4})^{\delta}} ,$$

and matching these two forms gives $\delta=\frac{4}{3}$. Interestingly, this value for $\delta$ is the same as the corresponding exponent in the Matheron and de Marsily model, although the subset of walks that contribute to the large-distance decay in the two cases are different. In random shear, transversely stretched walks determine the large-distance tail of the probability distribution, while in the Matheron and de Marsily model, the relevant contribution is due to transversely compressed walks.

## IV. SUMMARY

We have introduced a simple transport model which gives rise to superballistic motion. In our model a diffusing particle is driven by a steady two-dimensional shear flow, with the magnitude of the velocity generated by a random-walk process. This flow leads to the longitudinal displacement of the particle growing as $t^{5/4}$, i.e., faster than linearly in time. One of the more striking aspects of this model is that the probability distribution of longitudinal displacements if bimodal for a single typical configuration of the flow, while the configuration average of the distribution is unimodal. The average distribution exhibits a sharp cusp at the origin, which stems from a logarithmic singularity in the underlying scaling function, and an anomalously slow large-distance decay. This slow decay stems from the small subset of transversely stretched walks in a linear shear configuration, for which the longitudinal displacement grows as $t^2$. The large-distance behavior of the probability distribution is masked by a slow crossover in which the estimate for the shape exponent $\delta$ is a gradually increasing function of the order of the moment being analyzed. As this moment order increases, there is a systematic trend in our estimate for the shape exponent toward $\delta=\frac{4}{3}$. The many peculiar features of the probability distribution appear to stem from long-range correlations induced by the long periods of time that the random walk spends in regions with $v_x(y(t))>0$ and in regions with $v_x(y(t))<0$, an issue which merits further investigation.

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