

Constrained Opinion Dynamics: Freezing and Slow Evolution

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We study opinion formation in a population of leftists, centrists, and rightist. In an interaction between neighboring agents, a centrist and a leftist can become both centrists or leftists (and similarly for a centrist and a rightist), while leftists and rightists do not affect each other. For any spatial dimension the final state is either consensus (of one of three possible opinions), or a frozen population of leftists and rightists. In one dimension, the opinion evolution is mapped onto a constrained spin-1 Ising model with zero-temperature Glauber kinetics. The approach to the final state is governed by a $t^{-\psi}$ long-time tail, with ψ a non-universal exponent that depends on the initial densities. In the frozen state, the length distribution of single-opinion domains has an algebraic small-size tail $x^{-2(1-\psi)}$ with average domain length $L^{2\psi}$, where L is the length of the system.

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One of the basic issues in opinion dynamics is to understand the conditions under which consensus or diversity is reached from an initial population of individuals (agents) with different opinions. Models for such evolution are typically based on each agent freely adopting a new state in response to opinions in a local neighborhood [1]. The attribute of incompatibility – in which agents with sufficiently disparate opinions do not interact – has recently been found to prevent ultimate consensus from being reached [2, 3]. Related phenomenology arises in the Axelrod model [4, 5], a simple model for the formation and evolution of cultural domains. The goal of the present paper is to investigate the role of incompatibility within a minimal model for opinion dynamics. This constraint has a profound effect on the nature of the final state. Moreover, there is anomalously slow relaxation to the final state. While we primarily frame our discussion in terms of opinion dynamics, our results apply equally well to the coarsening of spin systems. In the latter context, we obtain a new non-universal kinetic exponent in one dimension that originates from topological constraints on the arrangement of spins.

We consider a ternary system in which each agent can adopt the opinions of leftist, centrist, and rightist. The agents populate a lattice and in a single microscopic event an agent adopts the opinion of a randomly-chosen neighbor, but with the crucial proviso that that leftists and rightists are considered to be so incompatible that they do not interact. While a leftist cannot directly become a rightist (and vice versa) in a single step, the indirect evolution leftist \Rightarrow centrist \Rightarrow rightist is possible. Our model is similar to the classical voter model [6] and also turns out to be isomorphic to the 2-trait 2-state Axelrod model [4, 5]. Due to the incompatibility constraint in our

model, the final opinion outcome can be either consensus or a frozen mixture of extremists with no centrists. Figure 1 shows a typical frozen state on the square lattice (with periodic boundary conditions). Notice the nested enclaves of opposite opinions and the clearly visible clustering.

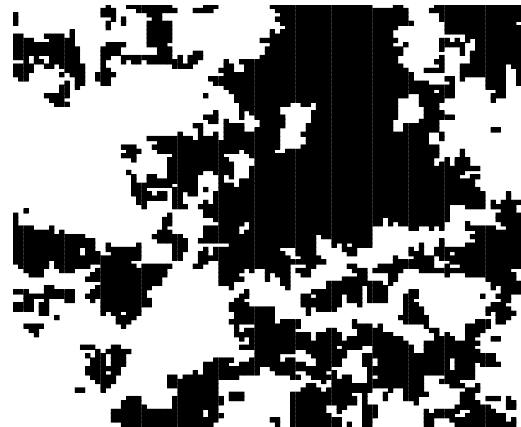


FIG. 1: Typical frozen final state in our opinion dynamics model on a 100×100 square lattice for $\rho_0 = 0.1$. The two extreme opinions are represented by black and white squares.

We can exploit the connection between the voter model and our opinion dynamics model to infer the final state of the latter. If we temporarily disregard the difference between leftists and rightists, the resulting binary system of centrists and extremists reduces to the voter model, for which one of two absorbing states — either all centrists or all extremists — is eventually reached. In the context of the ternary opinion system, the latter event can mean either a consensus of extremists or a frozen mixed state of leftists and rightists, as depicted in Fig. 1.

Because of the underlying voter model dynamics, the average density of each species is globally conserved in any spatial dimension. Therefore $\langle \rho_i(t) \rangle = \rho_i(t=0)$, where i refers to one of the states (+, 0, -) and the an-

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gle brackets denote an average over all realizations of the dynamics and over all initial states that are compatible with the specified densities. As a result of this conservation law, with probability $P_0 = \rho_0$ the final state consists of all centrists and with probability $1 - \rho_0$ there are no centrists in the final state. In the latter case, there can be either a consensus of $+$ (this occurs with probability P_+), consensus of $-$ (probability P_-), or a frozen mixed state (probability P_{+-}). Figure 2 shows the dependence of these final state probabilities on ρ_0 in the mean-field limit (where all agents are interconnected) in the symmetric case $\rho_+ = \rho_- = (1 - \rho_0)/2$. In one and two dimensions, the final state probabilities are nearly identical to the mean-field predictions when $\rho_+ = \rho_-$, but differences become apparent in the strongly asymmetric cases of $\rho_+ \gg \rho_-$. The final state probabilities also depend very weakly on the system size.

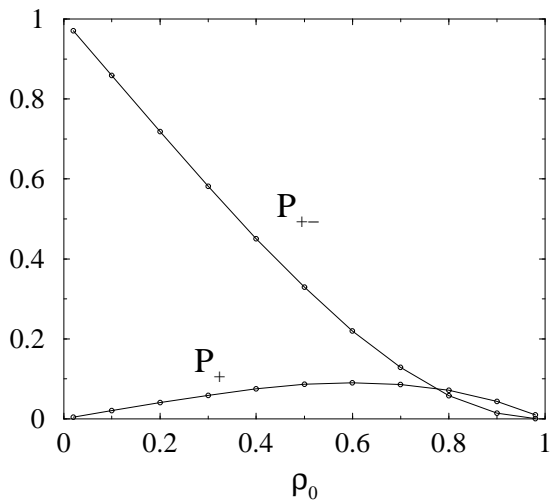
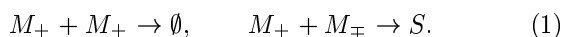


FIG. 2: Probability for the occurrence of a given final state as a function of ρ_0 for $\rho_+ = \rho_-$. Here P_+ is the probability that $+$ consensus is reached and P_{+-} is the probability that the final state is a frozen mixture of $+$ and $-$.

We now focus on the one dimensional case. Here our opinion dynamics model is equivalent to a constrained spin-1 Ising chain that is endowed with single-spin flip zero-temperature Glauber kinetics [7], with leftist, centrist, and rightist opinions equivalent to the spin states $-$, 0 , and $+$, respectively. The incompatibility constraint means that neighboring $+$ and $-$ spins do not interact.

This Ising model picture suggests that the best way to analyze the dynamics in one dimension is to reformulate the system in terms of domain walls. There are three types of domain walls: freely diffusing mobile domain walls between $+0$ and between -0 , denoted by M_+ and M_- , respectively, and stationary domain walls S between $+ -$. The mobile walls evolve by



When a mobile wall hits a stationary wall, the former changes its sign while the latter is eliminated via the

reaction



Thus stationary domain walls are dynamically invisible; their only effect is that the sign of a mobile wall changes whenever it meets a stationary wall, after which the latter disappears (Fig. 3). The inertness of the stationary walls is reminiscent of kinetic constraints in models of glassy relaxation. These constraints typically lead to extremely slow kinetics [8, 9, 10, 11], as is also observed in our opinion dynamics model.

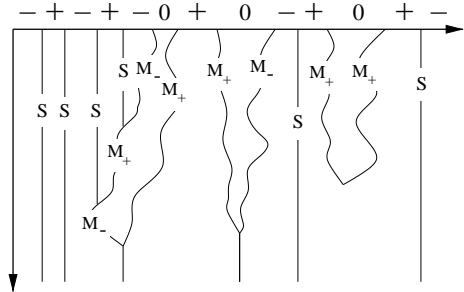


FIG. 3: Space-time representation of the domain wall dynamics. Time runs vertically downward. The spin state of the domains and the identity of each domain wall are indicated.

The rate equations corresponding to the processes in Eqs. (1) and (2) are

$$\dot{M} = -2M^2 \quad \dot{S} = -MS + M^2. \quad (3)$$

These give the asymptotic behaviors $M \sim (2t)^{-1}$ and $S \propto t^{-1/2}$. Thus an approximate rate equation approach already predicts that stationary domain walls decay slower than mobile walls.

An important subtlety in the arrangement of domain walls is that an arbitrary initial opinion state necessarily leads to an *even* number of mobile walls between each pair of stationary walls. It is also easy to verify that domain wall sequences of the form $\dots M_+ M_- M_+ \dots$ cannot arise from an underlying opinion state. These topological constraints play a crucial role in the kinetics.

The exact density of mobile walls can be obtained by again mapping the constrained spin-1 system onto a spin-1/2 system that is equivalent to the voter model. In this mapping, we consider both $+$ and $-$ spins as comprising the same (non-zero) spin state, while the zero spins comprise the other state. With this identification, the reduced model is just the spin-1/2 ferromagnetic Ising chain with zero-temperature Glauber dynamics and *no* kinetic constraint. In this reduced model, domain walls M_+ and M_- are indistinguishable and they diffuse and annihilate when upon colliding. The density of mobile walls $M(t) = M_+(t) + M_-(t)$ is known exactly for arbitrary initial conditions from the original Glauber solution [7]. For initially uncorrelated opinions and if the magnetization of the spin-1/2 system – here the difference

between the density of non-zero and zero spins – equals m_0 , then [12]

$$M(t) = \frac{1 - m_0^2}{2} e^{-2t} [I_0(2t) + I_1(2t)], \quad (4)$$

where I_k is the modified Bessel function of index k .

In the spin-1 system, $m_0 = \rho_+ + \rho_- - \rho_0$, or $m_0 = 1 - 2\rho_0$ by normalization. If the initial densities of + and – opinions are equal, then $M_+(t) = M_-(t)$ and their densities are

$$\begin{aligned} M_{\pm}(t) &= \rho_0(1 - \rho_0) e^{-2t} [I_0(2t) + I_1(2t)] \\ &\sim \rho_0(1 - \rho_0) (\pi t)^{-1/2}. \end{aligned} \quad (5)$$

As expected, the mobile wall density asymptotically decays as $t^{-1/2}$ because of the underlying diffusive dynamics. However, we find numerically that the density of stationary domain walls $S(t)$ decays as

$$S(t) \propto t^{-\psi(\rho_0)}, \quad (6)$$

with a *non-universal* exponent $\psi(\rho_0)$ that goes to zero as $\rho_0 \rightarrow 0$ (Fig. 4).

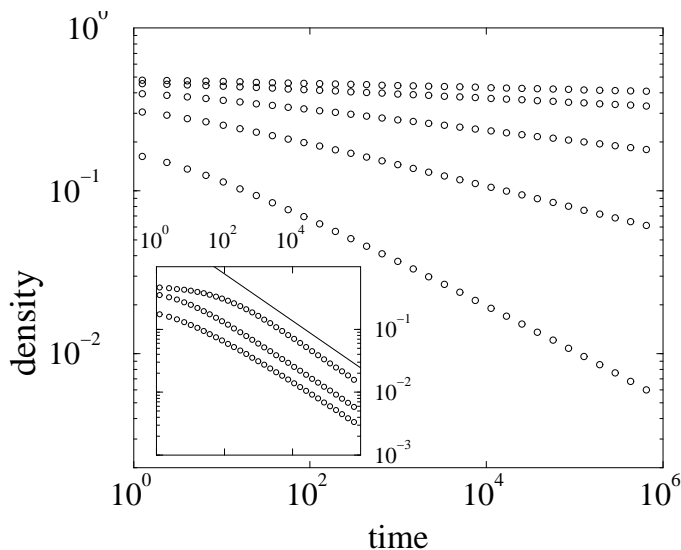


FIG. 4: Stationary domain walls density versus time on a double logarithmic scale for the initial conditions $\rho_0 = 0.02, 0.04, 0.10, 0.20,$ and 0.40 (top to bottom). The respective exponent estimates are $0.013, 0.026, 0.065, 0.13,$ and 0.29 . Data are based on 100 realizations on a 5×10^5 -site chain. Inset: Stationary domain wall density for initially uncorrelated walls for $\rho_0 = 0.02, 0.10,$ and 0.40 (top to bottom). The solid line has slope $-3/8$.

To help understand the mechanism for the slow decay of the stationary domain wall density, we also simulated a test system with spatially uncorrelated domain walls. While such a domain wall state cannot arise from any initial set of opinions, we can prepare directly an uncorrelated arrangement of domain walls with prescribed

densities. For any initial condition in this test system, the stationary wall density decays as $t^{-3/8}$ (inset to Fig. 4), consistent with known results on persistence [13, 14, 15]. Here persistence refers to the probability that a given lattice site is not hit by any diffusing domain wall. For the kinetic spin-1/2 Ising model, the persistence probability decays as $t^{-\theta}$, with $\theta = 3/8$ [16], independent of the initial domain wall density, when the walls are initially uncorrelated. Thus the topological constraints imposed on the domain wall arrangement by the initial opinion state appear to control the dynamics.

These topological constraints lead to the initial-condition dependence of the amplitude in the mobile wall density (Eq. (5)). This arises because for $\rho_0 \rightarrow 0$ the system initially consists of long strings of stationary walls that are interspersed by pairs of mobile walls, and their survival probability is proportional to their initial (unit) separation [17], leading to the asymptotic density for mobile walls is $M \sim 2\rho_0/\sqrt{\pi t}$ (Eq. (5)). We now exploit this observation to estimate the density of stationary walls as $\rho_0 \rightarrow 0$. Within a rate-equation approximation, the density of stationary domain walls decays according to

$$\dot{S} = -k M S. \quad (7)$$

While such an equation is generally inapplicable in low spatial dimension, we can adapt it to one dimension by employing an effective time-dependent reaction rate $k \sim \sqrt{2/\pi t}$ [14, 17]. This is just the time-dependent flux to an absorbing point due to a uniform initial background of diffusing particles; such a rate phenomenologically accounts for effects of spatial fluctuations in one dimension. Substituting the asymptotic expression for $M(t)$ from Eq. (4) and the reaction rate $k \sim \sqrt{2/\pi t}$ into this rate equation, we find that the density of stationary walls decays as $t^{-\psi}$ with $\psi(\rho_0) = \sqrt{8}\rho_0/\pi$ as $\rho_0 \rightarrow 0$. It is the amplitude in the density of mobile domain walls that ultimately causes the slow decay in the stationary wall density.

A more compelling way to determine $\psi(\rho_0)$ is via persistence in the q -state Potts model. Because the initial magnetization in the reduced spin-1/2 system is $m_0 = 1 - 2\rho_0$, it has been argued (see e.g. [18]) that this system should be identified with the q -state Potts model with $m_0 = 2/q - 1$, or $q = (1 - \rho_0)^{-1}$. Using the exact persistence exponent for the q -state Potts model with Glauber kinetics [16], and identifying ψ with this persistence exponent, we obtain

$$\psi(\rho_0) = -\frac{1}{8} + \frac{2}{\pi^2} \left[\cos^{-1} \left(\frac{1 - 2\rho_0}{\sqrt{2}} \right) \right]^2, \quad (8)$$

with the limiting behavior $\psi(\rho_0) \rightarrow 2\rho_0/\pi$ as $\rho_0 \rightarrow 0$. This asymptotics agrees with our numerical results for $\rho_0 \lesssim 0.4$ (Fig. 5) but then deviates for larger ρ_0 , where $\psi(\rho_0)$ must monotonically approach $1/2$ as $\rho_0 \rightarrow 1$. It should be noted that in this identification with persistence, we have ignored the creation of stationary interfaces due to the meeting of mobile domain walls. This

creation process occurs with a rate $(-dS/dt)_{\text{gain}} \propto t^{-3/2}$ and is subdominant for $\psi < 1/2$ [19].

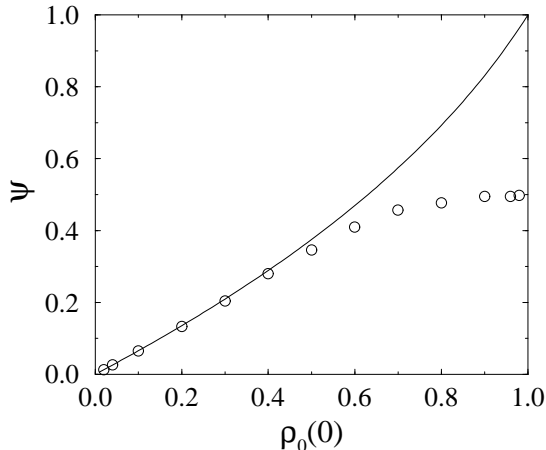


FIG. 5: Comparison of the exponent ψ from Eq. (8) and from the simulation data of Fig. 4 (circles).

An important characteristic of the system is the spatial arrangement of domain walls. From our simulations, the mean distances between nearest-neighbor MM and MS walls both appear to grow as $t^{1/2}$ due to the diffusive motion of mobile domain walls. The distributions of these two distances both obey scaling, with scaling function of the form ze^{-z^2} , where $z = x(t)/\langle x(t) \rangle$ is the scaled separation between walls. In contrast, the distances between neighboring stationary walls x_{SS} appear to be characterized by two length scales. There are large gaps of length of the order of $t^{1/2}$ that are cleared out by mobile walls before they annihilate, but there are also many very short distances remaining from the initial state (Fig. 3). The corresponding moments $\langle x_{SS}^k(t) \rangle^{1/k}$ reflect this multiplicity of scales, with $\langle x_{SS}^k(t) \rangle^{1/k}$ approaching a $t^{1/2}$ growth law as $k \rightarrow \infty$, and growing extremely slowly in time for $k \rightarrow 0$.

Finally, we quantify the frozen final state by the magnetization distribution $P(m)$, namely, the density difference between $+$ and $-$ spins. This distribution becomes broader as ρ_0 increases (Fig. 6), reflecting the fact that there is progressively more evolution before the system ultimately freezes. For small ρ_0 , $P(m)$ has a m^{-2} tail. We may explain this result by considering the evolution of a single pair of mobile walls. This pair annihilates at time t with probability density $\Pi(t) \propto t^{-3/2}$. The total magnetization of the resulting frozen state scales as $t^{1/2}$ since the domain wall pair annihilates at a distance $x \propto t^{1/2}$ from its starting point. Then from $P(m) dm = \Pi(t) dt$, together with $\Pi(t) \propto t^{-3/2}$ and $m \propto \rho_0 t^{1/2}$, we obtain $P(m) \propto \rho_0 m^{-2}$. While the argument has been formulated in one dimension, we expect this result to apply in all spatial dimension.

The frozen state is reached when $t = T_f \propto L^2$; this is the time needed for mobile domain walls to diffuse throughout the system and thus be eliminated. At this

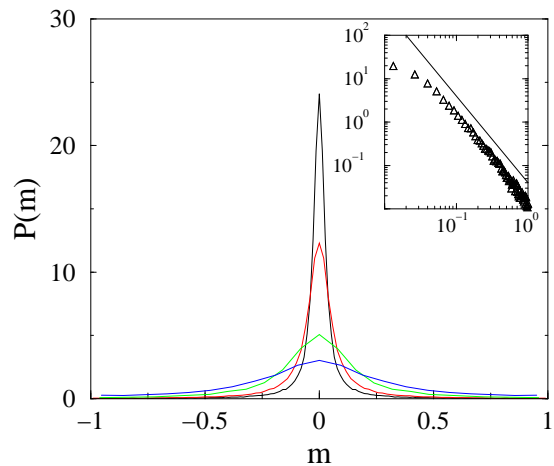


FIG. 6: Magnetization distribution $P(m)$, ($m = \rho_+ - \rho_-$) in the frozen final state for $\rho_0 = 0.02$ (10^5 realizations), 0.04, 0.1, and 0.2 (10^4 realizations) on a 5000 site linear chain. Inset: $P(m)$ for $\rho_0 = 0.02$ on a double logarithmic scale to illustrate the m^{-2} tail. The straight line has slope -2 .

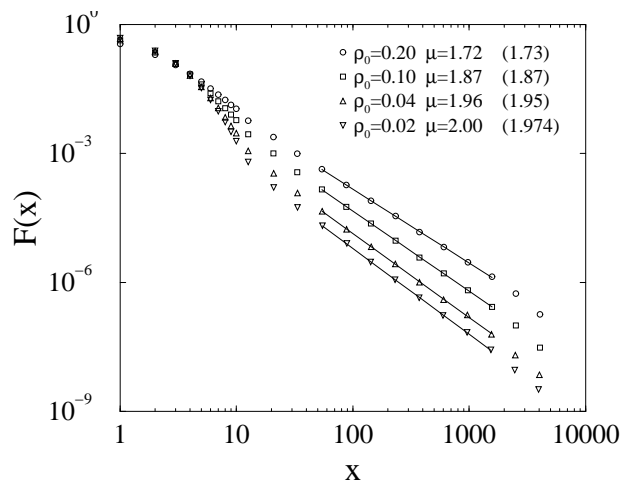


FIG. 7: Domain length distribution $F(x)$ in the frozen final state for the same data in Fig. 6. The data have been binned over a small logarithmic range to reduce fluctuations. Tabulated are the slopes from each data set and the expected value $2(1 - \psi)$ (parentheses) from our simulation result for ψ .

time, the density of stationary domain walls is of the order of $S \propto T_f^{-\psi} \propto L^{-2\psi}$. Thus the average length of single-opinion domains is $\langle x \rangle \propto L^{2\psi}$. Numerically we find that the domain length distribution has a power law tail, $F(x) \propto x^{-\mu}$, with $1 < \mu < 2$. The lower bound ensures normalizability while the upper bound implies that $\langle x \rangle$ diverges as $L \rightarrow \infty$. From the above power law form, $\langle x \rangle = \int dx x F(x) \propto L^{2-\mu}$. This matches our previous estimate of $\langle x \rangle \sim L^{2\psi}$ when $\mu = 2(1 - \psi)$. Figure 7 shows the length distribution of single-opinion domains in the frozen state. Direct estimates of the exponent μ

from this plot are in good agreement with the exponent relation $\mu = 2(1 - \psi)$, with ψ obtained from the time dependence of the mobile wall density in Fig. 4.

In summary, the constraint that extremists with opposite opinions cannot influence each other substantially slows down opinion dynamics. In one dimension, the density of stationary interfaces between neighboring + and - spins decays as $t^{-\psi}$, with $\psi(\rho_0) \sim 2\rho_0/\pi$ as $\rho_0 \rightarrow 0$. This slow dynamics is a consequence of the subtle topological constraints on the arrangement of domain walls, together with the incompatibility constraint that neighboring + and - spins do not interact.

The final opinion outcome depends non-trivially on the initial densities of the leftist, rightist and centrist opinion states. With probability ρ_0 the final population consists

of only centrists, while with probability $1 - \rho_0$ the final population does not contain any centrists. Analytically, we only know that consensus of centrists is reached with probability $P_0 = \rho_0$; the determination of the complementary final state probabilities for general initial conditions P_+ , P_- , and P_{+-} remains an open challenge. Finally, a frozen mixture final state would not exist if there is a non-zero rate (however small) for opposite extremists to influence each other directly. Perhaps this lack of direct influence can account for the sad phenomenon of the proximity of incompatible populations in too many locations around the world.

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