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LETTER TO THE EDITOR

Constrained opinion dynamics: freezing and slow evolution

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Abstract
We study opinion formation in a population of leftists, centrists and rightist. In an interaction between neighbouring agents, a centrist and a leftist can become both centrists or leftists (and similarly for a centrist and a rightist), while leftists and rightists do not affect each other. The evolution is controlled by the initial density of centrists ρ_0. For any spatial dimension the system reaches a centrist consensus with probability ρ_0, while with probability $1-\rho_0$ the final state is either an extremist consensus, or a frozen population of leftists and rightists. In one dimension, we determine the opinion evolution by mapping the system onto a spin-1 Ising model with zero-temperature Glauber kinetics. The approach to the final state is governed by a $t^{-\psi}$ long-time tail, with $\psi \to 2\rho_0/\pi$ as $\rho_0 \to 0$. In the one-dimensional frozen state, the length distribution of single-opinion domains has an algebraic small-size tail $x^{-2(1-\psi)}$ and the average domain length is $L^{2\psi}$, where L is the length of the system.

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One of the basic issues in opinion dynamics is to understand the conditions under which consensus or diversity is reached from an initial population of individuals (agents) with different opinions. Models for such evolution are typically based on each agent freely adopting a new state in response to the opinions in a local neighbourhood [1]. The attribute of incompatibility—in which agents with sufficiently disparate opinions do not interact—has recently been found to prevent ultimate consensus from being reached [2, 3]. Related phenomenology arises in the Axelrod model [4, 5], a simple model that accounts for the formation and evolution of cultural domains. The goal of the present letter is to investigate the role of the incompatibility constraint on opinion formation within a minimal model for which quantitative results can be obtained. We find that the incompatibility restriction has a profound effect on the dynamics and on the nature of the final state.

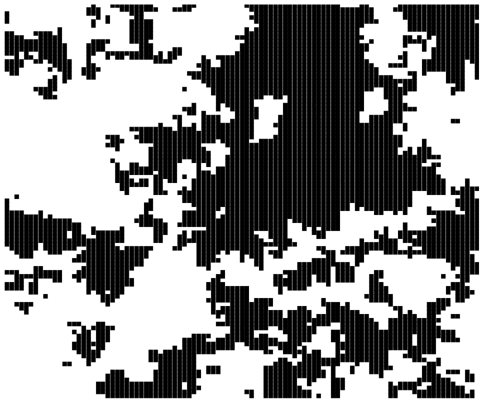

Figure 1. Typical frozen final state in our opinion dynamics model on a 100×100 square lattice for $\rho_0 = 0.1$. The two extreme opinions are represented by black and white squares.

We consider a ternary system in which each agent can adopt the opinions of leftist, centrist and rightist. The agents populate a lattice and in a single microscopic event an agent adopts the opinion of a randomly-chosen neighbour, but with the crucial proviso that leftists and rightists are considered to be so incompatible that they do not interact. Note that while a leftist cannot directly become a rightist (and vice versa) in a single step, the evolution leftist $\Rightarrow$ centrist $\Rightarrow$ rightist is possible. Our model is similar to the classical voter model [6] and also turns out to be isomorphic to the two-trait two-state Axelrod model [4, 5]. Due to the incompatibility constraint in our model, the final opinion outcome can be either consensus or a frozen mixture of extremists with no centrists. Figure 1 shows a typical frozen state on the square lattice (with periodic boundary conditions). Note the existence of nested enclaves of opposite opinions and the clearly visible clustering.

We can exploit the connection between the voter model and our opinion dynamics model to infer the final state of the latter for any spatial dimension. If we temporarily disregard the difference between leftists and rightists, the resulting binary system of centrists and extremists reduces to the voter model for which one of the two absorbing states of all centrists or all extremists is eventually reached. In the context of the ternary opinion system, the latter case can mean either a consensus of extremists or a frozen mixed state of leftists and rightists as depicted in figure 1.

Because of the underlying voter model dynamics, the average density of each species is globally conserved in any spatial dimension. That is, $\langle \rho_i(t) \rangle = \rho_i(t = 0)$, where i refers to one of the states $(+, 0, -)$ and the angle brackets denote an average over all realizations of the dynamics and over all initial states that are compatible with the specified density. Thus with probability $P_0 = \rho_0$ the final state consists of all centrists, while with probability $1 - \rho_0$ the final state consists of no centrists. In the latter case, there can be either a consensus of + (this occurs with probability P_+), consensus of $-$ (probability P_-), or a frozen mixed state (probability P_{+-}). Figure 2 shows these final-state probabilities as a function of the initial densities. Because of the conservation of the global density in the model, this behaviour is valid for any spatial dimension. In principle, the final-state probabilities could depend on the system size, but this dependence turns out to be extremely weak.

To understand the approach to the final state, we now focus on the case of one dimension. Here our opinion dynamics model is equivalent to a constrained spin-1 Ising chain that is endowed with single-spin flip zero-temperature Glauber kinetics [7], with leftist, centrist and

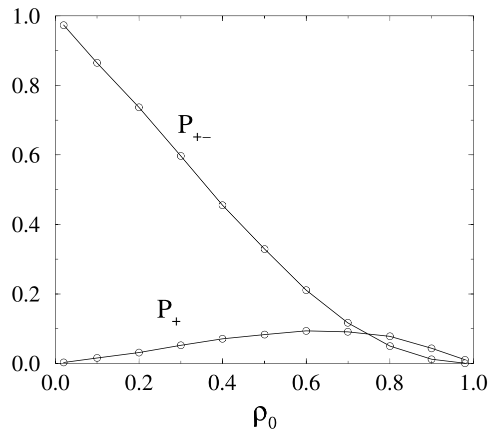

Figure 2. Probability for the occurrence of a given final state as a function of ρ_0 for $\rho_+ = \rho_-$. Here P_+ is the probability that + consensus is reached and P_{+-} is the probability that the final state is a frozen mixture of + and −.

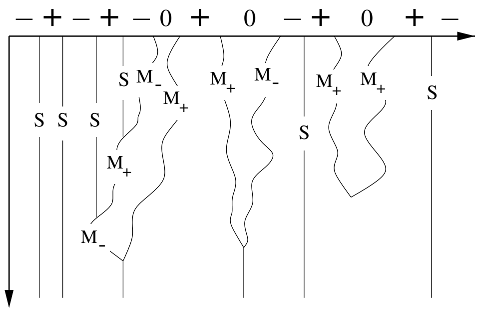

Figure 3. Space–time representation of the domain wall dynamics. Time runs vertically downward. The spin state of the domains and the identity of each domain wall are indicated.

rightist opinions equivalent to the spin states −, 0 and +. The incompatibility constraint is simply that neighbouring + and − spins do not interact.

This Ising model picture suggests that the best way to analyse the dynamics in one dimension is to reformulate the system in terms of domain walls. There are three types of domain walls: freely diffusing mobile domain walls between +0 and −0, denoted by M_+ and M_-, respectively, and stationary domain walls S between +−. The mobile walls evolve by

$$M_\pm + M_\pm \to \emptyset \qquad M_\pm + M_\mp \to S. \tag{1}$$

When a mobile wall hits a stationary wall, the former changes its sign while the latter is eliminated via the reaction

$$M_\pm + S \to M_\mp. \tag{2}$$

Thus stationary domain walls are dynamically invisible; their only effect is that the sign of a mobile wall changes whenever it meets a stationary wall, after which the latter disappears (figure 3). The inertness of the stationary walls is reminiscent of kinetic constraints in models of glassy relaxation. These constraints typically lead to extremely slow kinetics [8–11]; this feature also occurs in our opinion dynamics model.

The rate equations corresponding to the processes in equations (1) and (2) are

$$\dot{M} = -2M^2 \qquad \dot{S} = -MS + M^2. \tag{3}$$

These give the asymptotic behaviours $M \sim (2t)^{-1}$ and $S \propto t^{-1/2}$. Thus in the mean-field limit, stationary domain walls decay much slower than mobile walls. This distinction between stationary and mobile domain walls turns out to be even more pronounced in one dimension.

An important subtlety in the arrangement of domain walls is that an arbitrary initial opinion state necessarily leads to an *even* number of mobile walls between each pair of stationary walls. It is also easy to verify that domain wall sequences of the form $\ldots M_+M_-M_+\ldots$ cannot arise from an underlying opinion state. These topological constraints play a crucial role in the kinetics.

The exact density of mobile walls can be obtained by mapping the constrained spin-1 system onto a spin-1/2 system; this is equivalent to the voter model. In this mapping, we consider both + and − spins as comprising the same (non-zero) spin state, while the zero spins comprise the other state. With this identification, the reduced model is just the spin-1/2 ferromagnetic Ising chain with zero-temperature Glauber dynamics and *no* kinetic constraint. In the reduced model, domain walls M_+ and M_- are indistinguishable. These mobile walls undergo diffusion and annihilate upon colliding. Then the density of mobile walls $M(t) = M_+(t)+M_-(t)$ is known exactly for arbitrary initial conditions from the original Glauber solution [7]. For initially uncorrelated opinions and if the magnetization of the spin-1/2 system—here the difference between the density of non-zero and zero spins—equals m_0, then [12]

$$M(t) = \frac{1-m_0^2}{2}\,\mathrm{e}^{-2t}[I_0(2t) + I_1(2t)] \tag{4}$$

where I_k is the modified Bessel function of index k.

In our original spin-1 system, $m_0 = \rho_+ + \rho_- - \rho_0$, or $m_0 = 1 - 2\rho_0$ due to normalization. If the initial densities of + and − opinions are equal, then $M_+(t) = M_-(t)$ and the densities of mobile domain walls are

$$\begin{aligned} M_\pm(t) &= \rho_0(1-\rho_0)\,\mathrm{e}^{-2t}[I_0(2t) + I_1(2t)] \\ &\sim \rho_0(1-\rho_0)(\pi t)^{-1/2}. \end{aligned} \tag{5}$$

As expected, the mobile wall density decays asymptotically as $t^{-1/2}$ because of the underlying diffusive dynamics. However, we find that the density of stationary domain walls $S(t)$ decays algebraically,

$$S(t) \propto t^{-\psi(\rho_0)} \tag{6}$$

with a *non-universal* exponent that goes to zero as $\rho_0 \to 0$ (figure 4).

To help understand the mechanism for the extremely slow decay of the stationary domain wall density, we also simulated a test system with spatially uncorrelated domain walls. While such a domain wall state cannot arise from any initial set of opinions, we can prepare directly an uncorrelated arrangement of domain walls with prescribed densities. For any initial condition in this test system, the stationary wall density decays as $t^{-3/8}$ (inset to figure 4), consistent with known results on persistence [13–15]. Here persistence refers to the probability that a given lattice site is not hit by any diffusing domain wall. For the kinetic spin-1/2 Ising model, this persistence probability decays as $t^{-\theta}$, with $\theta = 3/8$ [16], independent of the initial domain wall density, when the walls are initially uncorrelated. Thus the topological constraints imposed on the domain wall arrangement by the initial opinion state appear to control the dynamics.

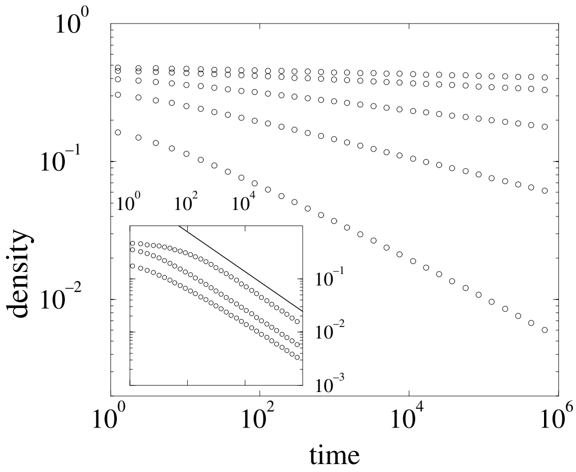

Figure 4. Density of stationary domain walls versus time on a double logarithmic scale for the initial conditions $\rho_0 = 0.02$, 0.04, 0.10, 0.20 and 0.40 (top to bottom). The respective exponent estimates are 0.013, 0.026, 0.065, 0.13 and 0.20. Data are based on 100 realizations of a chain with 5×10^5 sites. Inset: stationary domain wall density for initially uncorrelated walls for the same number of data. Shown are the cases $\rho_0 = 0.02$, 0.10 and 0.40 (top to bottom). The solid line has slope $-3/8$.

These topological constraints lead to a strong initial-condition dependence of the amplitude in the mobile wall density (equation (5)). This dependence arises because of the pairing of mobile domain walls in the initial state when ρ_0 is small. For $\rho_0 \to 0$, each pair becomes independent and their survival probability is proportional to their initial (unit) separation [17]. In contrast, the amplitude is independent of ρ_0 for randomly distributed walls. We now exploit this observation to estimate the density of stationary walls as $\rho_0 \to 0$. In this limit, the system initially consists of long strings of stationary domain walls that are interspersed by a pair of mobile domain walls. According to equation (5), the asymptotic density of mobile domain walls is $M \sim 2\rho_0/\sqrt{\pi t}$. Within a rate-equation approximation, the density of stationary domain walls decays according to

$$\dot{S} = -kMS. \tag{7}$$

While such a rate equation is generally inapplicable in low spatial dimension, we can adapt it to one dimension by employing an effective time-dependent reaction rate $k \sim \sqrt{2/\pi t}$ [14, 17]. This is just the time-dependent flux to an absorbing point due to a uniform initial background of diffusing particles; such a rate phenomenologically accounts for effects of spatial fluctuations in one dimension. Substituting the asymptotic expression for $M(t)$ from equation (4) and the reaction rate $k \sim \sqrt{2/\pi t}$ into this rate equation, we obtain the striking result that the density of stationary walls decays as $t^{-\psi}$ with $\psi(\rho_0) = \sqrt{8}\rho_0/\pi$ as $\rho_0 \to 0$. This qualitative argument shows how the amplitude in the density of mobile domain walls ultimately causes the slow decay in the stationary wall density.

A more compelling way to determine $\psi(\rho_0)$ is through a relation with persistence in the q-state Potts model. Because the initial magnetization in the reduced spin-1/2 system is $m_0 = 1 - 2\rho_0$, it has been argued (see [18]) that this system should be identified with the q-state Potts model with $m_0 = 2/q - 1$, or $q = (1 - \rho_0)^{-1}$. Using the exact persistence exponent for the q-state Potts model with Glauber kinetics [16], and identifying ψ with this persistence exponent, we obtain

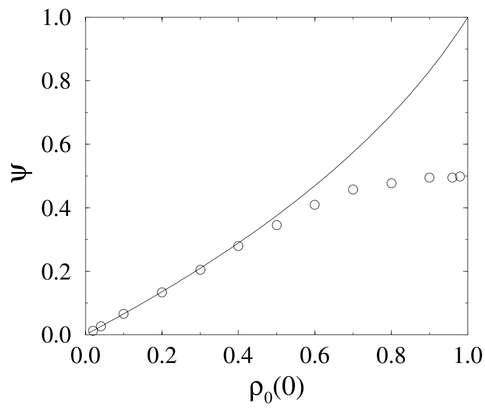

Figure 5. Comparison of the exponent ψ from equation (8) and from the simulation data of figure 4 (circles).

$$\psi(\rho_0) = -\frac{1}{8} + \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{1-2\rho_0}{\sqrt{2}}\right)\right]^2 \tag{8}$$

with the limiting behaviour $\psi(\rho_0) \to 2\rho_0/\pi$ as $\rho_0 \to 0$. This asymptotics agrees with our numerical results for $\rho_0 \lesssim 0.4$ (figure 5), but then deviates for larger ρ_0, where $\psi(\rho_0)$ must monotonically approach 1/2 as $\rho_0 \to 1$. It should be noted that in this identification with persistence, we have ignored the creation of new stationary interfaces due to the meeting of mobile domain walls M_+ and M_-. This creation process occurs with a rate $(-\mathrm{d}S/\mathrm{d}t)_{\text{gain}} \propto t^{-3/2}$. For $\psi < 1/2$, this creation term is subdominant and can be ignored [19].

An important characteristic of the system is the spatial arrangement of domain walls. From our simulations, the mean distances between nearest-neighbour MM and MS pairs both appear to increase as $t^{1/2}$ due to the diffusive motion of mobile domain walls. The distributions of these two distances both obey scaling, with scaling function of the form $z\,\mathrm{e}^{-z^2}$, where $z = x(t)/\langle x(t)\rangle$ is the scaled distance. In contrast, the distances between neighbouring stationary walls x_S appear to be characterized by two length scales. There are large gaps of length of the order of $t^{1/2}$ that are cleared out by mobile domain walls before they annihilate, but there are also many very short distances remaining from the initial state (figure 3). The corresponding moments $\langle x_S^k(t)\rangle^{1/k}$ reflect this multiplicity of scales, with $\langle x_S^k(t)\rangle^{1/k}$ approaching a $t^{1/2}$ growth law as $k \to \infty$, and growing extremely slowly in time for $k \to 0$.

Finally, we can quantify the frozen final state by the magnetization distribution $P(m)$, namely, the density difference between + and − spins. This distribution becomes broader as ρ_0 increases (figure 6), reflecting the fact that there is progressively more evolution before the system ultimately freezes. For small ρ_0, $P(m)$ has a m^{-2} tail. To explain this result, consider the evolution of a single pair of mobile walls. This pair annihilates at time t with probability density $\Pi(t) \propto t^{-3/2}$. The total magnetization of the resulting frozen state scales as $t^{1/2}$ since the domain wall pair annihilates at a distance $x \propto t^{1/2}$ from its starting point. Then from $P(m)\,\mathrm{d}m = \Pi(t)\,\mathrm{d}t$, together with $\Pi(t) \propto t^{-3/2}$ and $m \propto \rho_0 t^{1/2}$, we obtain $P(m) \propto \rho_0 m^{-2}$. While this argument has been formulated in one dimension, we expect the result to apply in all spatial dimensions because the final state probabilities are dimension independent.

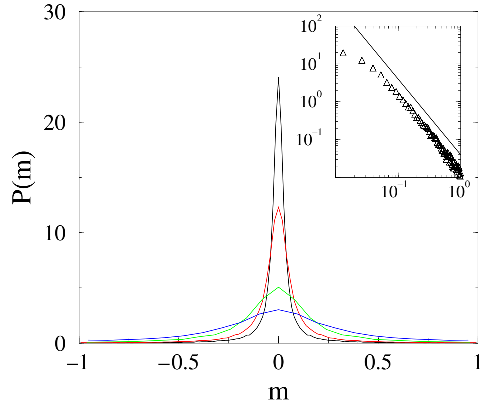

Figure 6. Magnetization distribution $P(m)$, with $m = \rho_+ - \rho_-$, in the frozen final state for the cases $\rho_0 = 0.02$ (10^5 realizations), 0.04, 0.1 and 0.2 (10^4 realizations) on a 5000 site linear chain. Inset: $P(m)$ for $\rho_0 = 0.02$ on a double logarithmic scale to illustrate the m^{-2} tail. The straight line has slope −2.

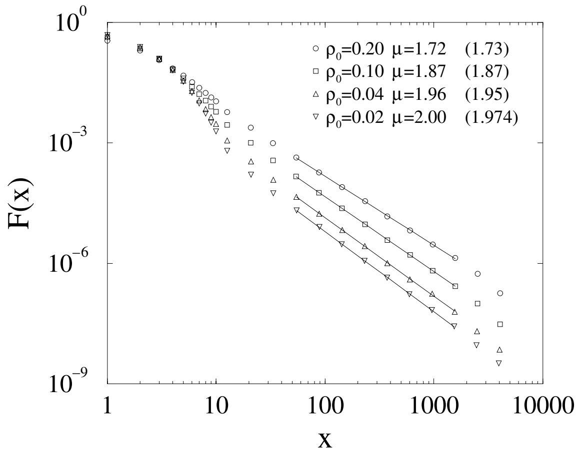

Figure 7. Domain length distribution $F(x)$ in the frozen final state for the same parameters and number of realizations as in figure 6. The data have been binned over a small logarithmic range to reduce fluctuations. Tabulated are the slopes from each data set and the expected value $2(1 - \psi)$ (parentheses) from our simulation result for ψ.

The frozen state is reached when $t = T_f \propto L^2$; this is the time needed for mobile walls to diffuse throughout the system and thus be eliminated. At this time, the density of stationary walls is of the order of $S \propto T_f^{-\psi} \propto L^{-2\psi}$. Thus the average length of single-opinion domains is $\langle x \rangle \propto L^{2\psi}$. Numerically, we find that the domain length distribution has a power-law tail, $F(x) \propto x^{-\mu}$, with $1 < \mu < 2$ also dependent on ρ_0. The lower bound ensures normalizability while the upper bound implies that $\langle x \rangle$ diverges as $L \to \infty$. From the above power-law form, $\langle x \rangle = \int \mathrm{d}x\, x\, F(x) \propto L^{2-\mu}$. This matches our previous estimate of $\langle x \rangle \sim L^{2\psi}$ when $\mu = 2(1 - \psi)$. Figure 7 shows the length distribution of single-opinion domains in the frozen state. Direct estimates of the exponent μ from this plot are in good agreement with the exponent relation $\mu = 2(1 - \psi)$, with ψ obtained from the time dependence of the mobile wall density in figure 4.

In summary, the constraint that extremists with opposite opinions cannot influence each other leads to an extremely slow opinion dynamics and a rich behaviour for the selection of the final state. In one dimension, the density of stationary interfaces between neighbouring + and − spins decays as $t^{-\psi}$, with $\psi(\rho_0) \sim 2\rho_0/\pi$ as $\rho_0 \to 0$. This slow dynamics is a consequence of the subtle topological constraints on the arrangement of domain walls, together with the incompatibility constraint that neighbouring + and − spins do not interact. In the mean-field limit, the densities of mobile and stationary domains decay as t^{-1} and $t^{-1/2}$, respectively. We expect that these predictions should also apply in two dimensions.

The composition of the final state also has a non-trivial dependence on the initial densities of the leftist, rightist and centrist opinion states. With probability ρ_0 the final population consists of only centrists, while with probability $1 - \rho_0$ the final population does not contain any centrists. In this latter case, there can be either a consensus of extremists or the final state can consist of a frozen mixture of extremist enclaves. If there is a bias in the initial state, that is, $\rho_+/\rho_- > 1$, then it becomes more likely that consensus of + will be achieved. However, there is still a substantial probability that the system still ends up as a frozen mixture of + and −. Finally, a frozen mixture final state would not exist if there is a non-zero rate (however small) for opposite extremists to influence each other directly. Perhaps this lack of direct influence can account for the sad phenomenon of the proximity of incompatible populations in too many locations around the world.

Acknowledgments

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