Dynamics of Non-Conservative Voters

R. Lambiotte[1] and S. Redner[2]
[1] *INMA, Université catholique de Louvain, 4 avenue Georges Lemaitre, B-1348 Louvain-la-Neuve, Belgium*
[2] *Center for Polymer Studies and Physics Department, Boston University, Boston, MA 02215 USA*

We study a family of opinion formation models in one dimension where the propensity for a voter to align with its local environment depends non-linearly on the fraction of disagreeing neighbors. Depending on this non-linearity in the voting rule, the population may exhibit a bias toward zero magnetization or toward consensus, and the average magnetization is generally not conserved. We use a decoupling approximation to truncate the equation hierarchy for multi-point spin correlations and thereby derive the probability to reach a final state of ↑ consensus as a function of the initial magnetization. The case when voters are influenced by more distant voters is also considered by investigating the Sznajd model.

PACS numbers: 89.75.-k, 02.50.Le, 05.50.+q, 75.10.Hk

It often happens that individuals change their attitudes, behaviors and/or morals, to conform to those of their acquaintances. Perhaps the simplest description of this conformity is the voter model [1], where each node of a graph (*i.e.*, the social network) is occupied by a voter that has one of two opinions, ↑ or ↓. In the voter model, the population evolves by: (i) picking a random voter; (ii) the selected voter adopts the state of a randomly-chosen neighbor; (iii) repeating these steps *ad infinitum* or until a finite system necessarily reaches consensus. Figuratively, voters have no self confidence and merely follow one of their neighbors. With this dynamics, a voter changes opinion with a probability p_f that equals the fraction f of disagreeing neighbors. This proportionality rule leads to the conservation of the average opinion in the system, a feature that renders the voter model soluble in all dimensions [1, 2]. However, the specific rule $p_f = f$ is one among many possible and socially plausible relations between p_f and f.

In this work, we generalize the voter model so that p_f depends non-linearly on f. Non-linear voter models have been discussed previously, primarily by numerical simulations in two dimensions [3]. Here we focus on one dimension, where the range of possibilities for the non-linearity is limited. In one dimension, a voter may be confronted by 0, 1, or 2 disagreeing neighbors. It is natural to impose $p_0 = 0$, so that no evolution occurs when there is local consensus. Then the most general description of the system requires two parameters, p_1 and p_2 (Fig. 1). One parameter, which we choose to be p_1, determines the overall time scale and is thus immaterial. The only relevant parameter then is $\gamma = p_2/p_1$. When $\gamma = 2$, one recovers the classical voter model. When $\gamma > 2$, the combined effect of two neighbors is more than twice that of one neighbor. Equivalently, voters can be viewed as having a conviction for their opinion and that strong peer pressure is needed to change opinion. As $\gamma \to \infty$, voters only change opinion when are confronted by a unanimity of opposite-opinion voters [4]. In contrast, when $\gamma < 2$, one disagreeing neighbor is more effective in triggering an opinion change than in the classical voter model. When $\gamma = 1$, one recovers the *vacillating* voter model [5, 6] where voters change opinion at a fixed rate if either 1 or 2 of their neighbors disagree with them. Finally, $\gamma < 1$ corresponds to a "contrarian" regime where a voter is less likely to change opinion as the fraction of neighbors in disagreement increases.

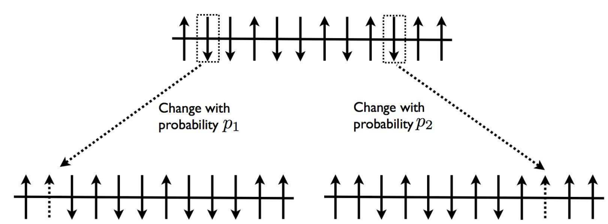

FIG. 1: Update illustration. A random voter changes state with probability p_1 if it has 1 disagreeing neighbor (left), and with probability p_2 if it has 2 disagreeing neighbors (right).

We term γ the conviction parameter and view voters as having or lacking conviction if $\gamma > 2$ or $\gamma < 2$, respectively. When $\gamma > 2$, the population is quickly driven to consensus, while when $\gamma < 2$, there is a bias toward the zero-magnetization state with equal densities of voters of each type. In the former case, the collective evolution is more extreme than the evolution of an individual. In the latter case, collective evolution is hindered and the approach to final consensus is anomalously slow. In both cases, we can determine the *exit probability*, namely, the probability the system ultimately reaches ↑ consensus as a function of the initial composition of the population.

In the framework of the Ising-Glauber model [7], the transition rate of a voter at site i, whose states we now represent by $\sigma_i = \pm 1$, is

$$w(\{\sigma\}\to\{\sigma'\}_i) = \gamma + 2 - \gamma\sigma_i(\sigma_{i-1} + \gamma\sigma_{i+1}) + (\gamma - 2)\sigma_{i-1}\sigma_{i+1}, \qquad (1)$$

with $\{\sigma\}$ denoting the state of all voters, $\{\sigma'\}_i$ the state where the i^{th} voter has flipped, and where an irrelevant overall constant has been absorbed in the time scale. The fact that the $\sigma_i\sigma_{i+1}$ and the $\sigma_i\sigma_{i-1}$ terms have the same coefficient is due to the left/right symmetry of the dynamics. Note also that for $\gamma = 2$, the $\sigma_{i-1}\sigma_{i+1}$ term vanishes, and the equation of motion of the classic voter model is recovered. As shown in Glauber's original work [7] and as we show below, the $\sigma_{i-1}\sigma_{i+1}$ term couples the rate equation for the mean spin to 3-body terms.

Because the average magnetization is not conserved, the model is not exactly soluble. Nevertheless, we can construct an approximate solution by truncating the hierarchy of rate equations of higher-order correlation functions by a simple decoupling scheme. The mean spin, $s_j \equiv \langle\sigma_j\rangle = \sum_{\{\sigma\}} \sigma_j P(\{\sigma\};t)$ evolves as

$$\frac{\partial s_j}{\partial t} = \sum_{\{\sigma\}} \sigma_j \Big[\sum_i w(\{\sigma'\}_i \to \{\sigma\})\, P(\{\sigma'\}_i;t) - w(\{\sigma\} \to \{\sigma'\}_i)\, P(\{\sigma\};t) \Big]. \quad (2)$$

After straightforward steps, Eq. (2) reduces to

$$\frac{\partial s_j}{\partial t} = 2\gamma(s_{j+1} + s_{j-1}) - 2(\gamma+2)s_j - 2(\gamma-2)\langle\sigma_{j-1}\sigma_j\sigma_{j+1}\rangle, \quad (3)$$

which depends only on the mean spins s_j, s_{j-1} and s_{j+1} when $\gamma = 2$, as expected, but is otherwise coupled to higher order correlations.

Let us first consider the mean-field limit, where the spins of neighboring nodes are uncorrelated. Because of spatially homogeneity, $\langle s_j\rangle$ are all identical and we write the magnetization as $m \equiv \langle s_j\rangle$. Then assuming that $\langle\sigma_{j-1}\sigma_j\sigma_{j+1}\rangle \approx m^3$, Eq. (3) simplifies to

$$\frac{\partial m}{\partial t} = 2(\gamma-2)(m - m^3), \quad (4)$$

which shows that the magnetization is not conserved when $\gamma \neq 2$. The stable solutions of Eq. (4) are either consensus ($m = \pm 1$) when $\gamma > 2$, or stasis, with equal densities of the two types of voters ($m = 0$), when $\gamma < 2$.

In one dimension, however, the mean-field assumption is not justified. To determine the behavior of the system when $\gamma \neq 2$, we therefore use the approach developed in [5] (see also [8]) to truncate the hierarchy of equations for multi-spin correlation functions. Consider the rate equation for the nearest-neighbor correlation function $\langle\sigma_j\sigma_{j+1}\rangle$:

$$\frac{\partial\langle\sigma_j\sigma_{j+1}\rangle}{\partial t} = -2(\gamma-2)\left[\langle\sigma_{j-1}\sigma_j\rangle + \langle\sigma_{j+1}\sigma_{j+2}\rangle\right] + 2\gamma\left[\langle\sigma_{j-1}\sigma_{j+1}\rangle + \langle\sigma_j\sigma_{j+2}\rangle\right] + 4\gamma - 4(\gamma+2)\langle\sigma_j\sigma_{j+1}\rangle. \quad (5)$$

To close this equation, we need to approximate the second-neighbor correlation function $\langle\sigma_j\sigma_{j+2}\rangle$. Consider domain walls—nearest-neighbor anti-aligned voters—whose density is $\rho = (1 - \langle\sigma_i\sigma_{i+1}\rangle)/2$. According to the transition rate in Eq. (1), an isolated domain wall diffuses freely for any γ. However, when two domain walls are adjacent, they annihilate with probability $P_{\rm a} = \gamma/(2+\gamma)$ or they hop away from each other with probability $P_{\rm h} = 2/(2+\gamma)$. Thus when $\gamma > 2$, $P_{\rm a} > P_{\rm h}$ and adjacent domain walls have a tendency to annihilate, while they are repelled from each other when $\gamma < 2$. The interaction of two domain walls is therefore equivalent to single-species annihilation, $A + A \to 0$, but with a reaction rate that is modified compared to freely diffusing reactants because of this interaction. Nevertheless, the domain wall density asymptotically decays as $t^{-1/2}$ for any $\gamma < \infty$, and with an interaction-independent amplitude [9].

Since domain walls become widely separated at long times, we therefore approximate [5] the second-neighbor correlation function as $\langle\sigma_j\sigma_{j+2}\rangle \approx \langle\sigma_j\sigma_{j+1}\rangle$. We also define $m_2 \equiv \langle\sigma_j\sigma_{j+1}\rangle$ for a spatially homogeneous system. Now the rate equation (5) for the nearest-neighbor correlation function becomes

$$\frac{\partial m_2}{\partial t} = 4\gamma - 4\gamma m_2. \quad (6)$$

For the uncorrelated initial condition, $m_2(0) = m(0)^2$, the solution is

$$m_2(t) = 1 + \left[m(0)^2 - 1\right] e^{-4\gamma t}\ , \quad (7)$$

where $m(0) \equiv \langle s_j(0)\rangle$ is the average magnetization at $t = 0$.

In a similar spirit, we decouple the 3-spin correlation function $\langle\sigma_{j-1}\sigma_j\sigma_{j+1}\rangle \approx m m_2$ [5] and average over all sites, to simplify the rate equation (3) for the mean spin in a spatially homogeneous system to

$$\frac{\partial m}{\partial t} = 2(2-\gamma) m e^{-4\gamma t}(m(0)^2 - 1). \quad (8)$$

Solving this equation and taking the $t \to \infty$ limit, we obtain a non-trivial relation between the final magnetization $m(\infty)$ and the initial magnetization $m(0)$:

$$m(\infty) \ = \ m(0)\, e^{(2-\gamma)(m(0)^2-1)/2\gamma}\ . \quad (9)$$

It is important to realize that the average magnetization in a finite population does not perpetually fluctuate around this asymptotic value but rather ultimately reaches ± 1 because consensus is the only absorbing state of the stochastic dynamics. We characterize this approach to consensus by the exit probability $\mathcal{E}(x, N)$, defined as the probability that a population of N voters ultimately reaches $\uparrow$ consensus when there are initially $n = xN$ $\uparrow$ voters. Since the density of $\uparrow$ voters is $x = (1+m)/2$ and $m(\infty) = 2\mathcal{E}(x) - 1$, Eq. (9) leads to

$$\mathcal{E}(x) = \frac{1}{2}\left[(2x-1)e^{2x(2-\gamma)(x-1)/\gamma} + 1\right]\ . \quad (10)$$

The exit probability $\mathcal{E}(x)$ is independent of N, but has a non-trivial dependence on the initial condition. Similar behaviors have been previously found in other opinion

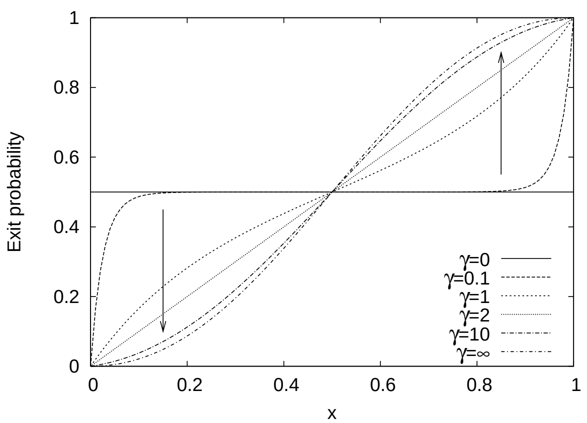

FIG. 2: Exit probability $\mathcal{E}(x)$ from Eq. (10) as a function of the initial density of ↑ voters x for different values of γ. The arrow indicates the position of the curves for increasing values of γ, namely from zero magnetization $\mathcal{E}(x) = 1/2$ ($\gamma = 0$) to more and more consensual systems.

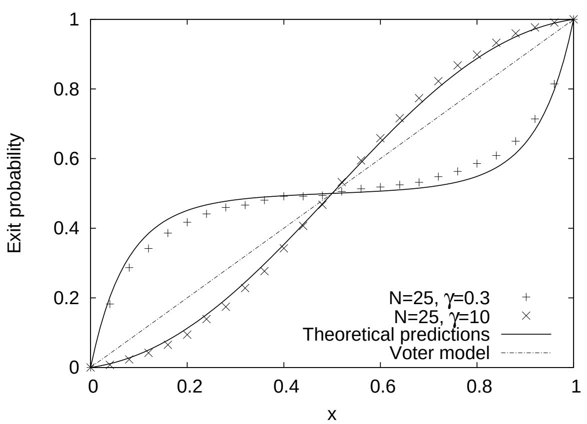

FIG. 3: Exit probability $\mathcal{E}(x)$ for a one dimensional system composed of 25 voters, for conviction parameter $\gamma = 0.3$ and 10.

evolution models where the average magnetization is not conserved, such as the majority rule model [10] and the Sznajd model [11].

The exit probability in (10) exhibits a qualitative change as γ passes through 2 (see Fig. 2). At $\gamma = 2$, the pure voter model result $\mathcal{E}(x) = x$, that follows from magnetization conservation, is recovered. When $\gamma < 2$, the dynamics drives the system toward zero magnetization before consensus is ultimately reached. Thus the exit probability becomes progressively more independent of the initial magnetization as $\gamma \to 0$. When $\gamma > 2$, in contrast, the system is driven toward consensus; in the $\gamma \to \infty$ limit, $\mathcal{E}(x)$ reduces to

$$\mathcal{E}(x) = \frac{1}{2}\left[(2x-1)e^{-2x(x-1)} + 1\right]. \qquad (11)$$

We checked the validity of Eq. (10) by simulations. To do so, we focused on small systems ($N = 25$) and directly measured the probability $\mathcal{E}(x)$ that the population ultimately reaches a ↑ consensus when the proportion of initially ↑ voters is x by averaging over 5000 realizations of the dynamics. The theoretical results are in excellent agreement with our simulation results (see Fig. 3).

It is in principle possible to generalize the model to higher dimensions d. However, in that case, each node is surrounded by $2d$ neighbors and $2d$ variables are therefore required, *i.e.*, the probabilities p_i that an individual with $i \in [1, 2d]$ disagreeing neighbors changes opinion, in order to specify completely the voters dynamics. Such a general analysis, that goes in the direction of the general model of contagion introduced by Dodds and Watts [12] and would include known models for opinion formation such as threshold models [13], will be considered in detail elsewhere. One should stress, however, that a much richer phenomenology can occur when the number of parameters p_i is increased [14]. In the case when each node is surrounded by four neighbors, for instance, and within a mean-field description, it is easy to show that the equation of evolution for the average density of ↑ voters is

$$\frac{\partial x}{\partial t} = \sum_{i=1}^{4}\binom{4}{i} p_i \left[(1-x)^{5-i}x^i - x^{5-i}(1-x)^i\right], \quad (12)$$

where we have again assumed that $p_0 = 0$. The stable stationary solutions of Eq. (12) are either consensus (when $p_1 < p_4/4$), a state of zero magnetization (when $-3p_1 - 3p_2/2 + p_3 + 3p_4/4 < 0$), or other asymmetric solutions, *e.g.*, when $p_1 = 1/2$, $p_0 = p_2 = 0$ and $p_3 = p_4 = 1$, the stable solutions are $x = (5 \pm \sqrt{5})/10$.

Let us now return to our analysis of one-dimensional systems. The approach developed above can also be applied to another simple opinion dynamics model in one dimension, namely, the Sznajd model [15]. This model is an appealing realization of the concept of social validation, namely, that agents are only influenced by groups (*e.g.*, pairs) of aligned voters and not by single individuals. The Sznajd is defined by the following evolution rule: (i) pick a pair of neighboring voters i and $i+1$; (ii) if these voters have the same opinion $\sigma_i = \sigma_{i+1}$, convert the opinion of the neighbors $i-1$ and $i+2$ on either side of the initial pair: $\sigma_{i-1} = \sigma_i = \sigma_{i+1} = \sigma_{i+2}$; (iii) repeat these steps *ad infinitum* or until a finite system necessarily reaches consensus.

It is straightforward to show that the transition rate of a voter at site i is

$$w(\{\sigma\}\to\{\sigma'\}_i) = -\frac{1}{4}\left[\sigma_i(\sigma_{i-2} + \sigma_{i-1} + \sigma_{i+1} + \sigma_{i+2}) - \sigma_{i-1}\sigma_{i-2} - \sigma_{i+1}\sigma_{i+2} - 2\right]; \quad (13)$$

notice that the rate for changing σ_i involves terms that include $\sigma_{i\pm2}$. Following the same steps as in our previous example of the voter model with variable conviction, the resulting equations for the average magnetization m and the nearest-neighbor correlation m_2 for a spatially homogeneous system are

$$\frac{\partial m}{\partial t} = m - m m_2\,, \qquad \frac{\partial m_2}{\partial t} = 1 - m_4\,, \qquad (14)$$

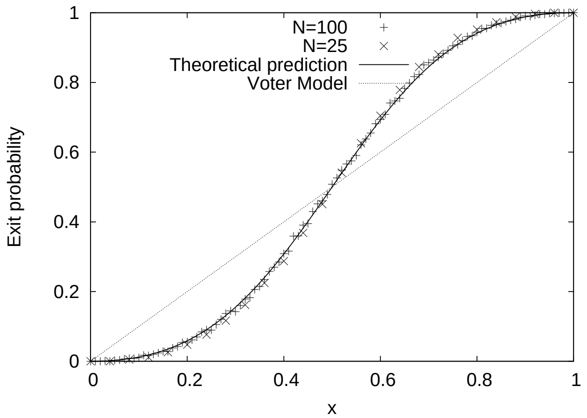

FIG. 4: Exit probability $\mathcal{E}(x)$ for the Sznajd model. The system is one dimensional and composed of 25 and 100 voters respectively. Results are averaged over 5000 realizations of the random process.

where we have again assumed that $\langle\sigma_j\sigma_{j+2}\rangle \approx \langle\sigma_j\sigma_{j+1}\rangle$ and $\langle\sigma_{j-1}\sigma_j\sigma_{j+1}\rangle \approx mm_2$. Here $m_4 \equiv \langle\sigma_{j-1}\sigma_j\sigma_{j+1}\sigma_{j+2}\rangle$ is the correlation between the states of four contiguous voters. This new term is due to the distant interactions between voters, and is also present in the majority rule model. Following [8], we apply the Kirkwood approximation to factorize the 4-point function as the product of 2-point functions, $m_4 \approx m_2^2$; this approach has proved quite useful in a variety of applications to reaction kinetics [16]. This truncation then allows us to solve the second of Eqs. (14) to give $m_2 = \frac{e^{2t}+C}{e^{2t}-C}$, where $C = (m(0)^2-1)/(m(0)^2+1)$, and finally to solve the equation for the average magnetization

$$m = \frac{e^{2t}}{e^{2t}-C}\,\frac{2m(0)}{1+m(0)^2}\ . \tag{15}$$

After some straightforward algebra, we then find the exit probability $\mathcal{E}(x)$

$$\mathcal{E}(x) = \frac{x^2}{1-2x+2x^2}\ . \tag{16}$$

One should stress that such a non-trivial exit probability has been observed previously in simulations of the Sznajd model [11], which clearly shows that the Sznajd model has a tendency toward consensus and that the prediction (16) is in remarkably good agreement with our simulations (see Fig. 4).

In this work, we investigated a general class of nonlinear voter models of opinion dynamics in one dimension. We considered the situation where the transition rate for each voter depends in a non-trivial way on the number of disagreeing neighbors. In general, the average magnetization is not conserved in these models and the evolution of the average opinion is coupled to higher-order opinion correlations. It is possible to truncate the hierarchy of equations for these correlations in a simple and plausible manner. From this approach, we find that the system coarsens, albeit differently than in the pure voter model because of the interactions (repulsion or attraction) of neighboring domain walls. The probability to reach the final state of $\uparrow$ consensus is also shown to have a non-trivial initial state dependence, a feature that reveals the tendency of the system to inhibit or to reach consensus. The decoupling approximation presented in this letter appears to be both efficient and robust, and therefore it could well be useful in other models of opinion formation, language dynamics, *etc.* [17], where the interactions between agents are not conservative.

Acknowledgments

We gratefully acknowledge the support of the ARC "Large Graphs and Networks" (RL), NSF grant DMR0535503 (SR), and the hospitality of the Ettore Majorana Center where this project was initiated.

Added Note: After this manuscript was completed, we became aware of parallel work by Slanina et al. [18], in which they obtain results similar to ours.

[1] T. M. Liggett, *Stochastic interacting systems: contact, voter, and exclusion processes*, (Springer-Verlag, New York, 1999).
[2] P. L. Krapivsky, Phys. Rev. A **45**, 1067 (1992).
[3] J. Molofsky, R. Durrett, J. Dushoff, D. Griffeath and S. Levin, *Theoretical Population Biology* **55**, 270 (1999)
[4] R. Lambiotte, S. Thurner and R. Hanel, Phys. Rev. E **76**, 046101 (2007)
[5] R. Lambiotte and S. Redner, J. Stat. Phys. L10001 (2007).
[6] For a related model, see also S. Morris, Rev. of Economic Studies **67**, 57 (2000) for a related model.
[7] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).
[8] M. Mobilia and S. Redner, Phys. Rev. E **68** 046106, (2003).
[9] D. ben-Avraham, in *Nonequilibrium Statistical Mechanics in One Dimension* chap. 2; ed. V. Privman (Cambridge University Press, Cambridge, 1997).
[10] S. Galam, J. Math. Psych. bf 30, 426 (1986); S. Galam, J. Stat. Phys. **61**, 943 (1990); P. L. Krapivsky and S. Redner, Phys. Rev. Lett. **90**, 238701 (2003).
[11] S. Krupa and K. Sznajd-Weron, Int. J. Mod. Phys. C **16**, 1771 (2005).
[12] P. S. Dodds and D. J. Watts, Phys. Rev. Lett. **92**, 218701 (2004)
[13] P. Klimek, R. Lambiotte and S. Thurner, arXiv:0706.4058.
[14] S. Galam, Europhys. Lett. **70**, 705 (2005)

[15] K. Sznajd-Weron and S. Krupa, Phys. Rev. E **74**, 031109 (2006).

[16] H. Schnörer, V. Kuzovkov and A. Blumen, Phys. Rev. Lett. **63** 805 (1989); J. C. Lin, C. R. Doering, and D. ben-Avraham, Chem. Phys. **146**, 355 (1990); L. Frachebourg and P. L. Krapivsky, J. Phys. A **31**, L287 (1998).

[17] C. Castellano, S. Fortunato, V. Loreto, Rev. Mod. Phys. **80**, xxx (2008).

[18] F. Slanina, K. Sznajd-Weron, P. Przybyla, arXiv:0712.2035v1 [cond-mat.stat-mech].