J. Phys. C: Solid State Phys., 16 (1983) 4339–4345. Printed in Great Britain

Break–collapse method for resistor networks and a renormalisation-group application

Constantino Tsallis†‖, Antonio Coniglio‡ and Sidney Redner§

† Centro Brasileiro de Pesquisas Fisicas/CNPq¶, Rua Xavier Sigaud 150, 22290 Rio de Janeiro, Brazil; and Department of Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK

‡ GNSM, Istituto di Física Teorica, Universitá di Napoli. Mostra d'Oltremare, Pad. 19, 80125 Napoli, Italy

§ Center for Polymer Studies* and Department of Physics, Boston University, Boston MA 02215, USA

Received 2 March 1983

Abstract. The break–collapse method recently introduced for the q-state Potts model is adapted for resistor networks. This method greatly simplifies the calculation of the conductance of an arbitrary two-terminal d-dimensional array of conductances, obviating the use of either Kirchhoff's laws or the star–triangle transformation. In addition, a real-space renormalisation group based on a new type of averaging gives excellent results for the conductivity of the random-resistor network on the square lattice.

1. Introduction

A considerable amount of effort is being devoted to the study of random resistor networks (see Deutscher 1981 and references therein for connections with experimental systems). A convenient way for quantitatively discussing such problems is through the real-space renormalisation group (RG) framework (Stinchcombe and Watson 1976, Straley 1977, Kirkpatrick 1977, Rosman and Shapiro 1977, Yeomans and Stinchcombe 1978, Bernasconi 1978, Kogut and Straley 1978, Lobb *et al* 1981, Mujeeb and Stinchcombe 1982). Typically within this procedure, the conductance of an arbitrary array of conductors has to be calculated. The purpose of the present work is to show that the break–collapse method (BCM), recently introduced for calculating analogous percolation, Ising and Potts arrays (Tsallis and Levy 1981), can be adapted to resistor networks, thus providing a simple way to calculate conductances. Our method avoids the use of Kirchhoff's laws, or the star–triangle transformation which becomes rather complicated for highly-coordinated lattices.

In § 2 we state the resistor BCM and related properties; in § 3 we illustrate its use within a RG calculation for the random resistor network on the square lattice. By using a new type of averaging procedure, excellent quantitative results are obtained.

‖ Guggenheim Fellow.

¶ Permanent address.

* Supported in part by grants from NSF, ONR and ARO.

2. Break–collapse method and related properties

Denote the *conductance* of a resistor as g. Then a parallel (series) array of two resistors g_1 and g_2 has a conductance of

$$g_p = g_1 + g_2 \qquad \text{parallel} \tag{1a}$$

$$g_s = g_1 g_2/(g_1 + g_2) \qquad \text{series.} \tag{1b}$$

The latter can be rewritten in a form analogous to (1*a*),

$$g_s^D = g_1^D + g_2^D \tag{1'}$$

with

$$g_i^D \equiv g_0^2/g_i \tag{2}$$

where D stands for *dual* (see Tsallis 1981, Alcaraz and Tsallis 1982 for a discussion in the context of the q-state Potts and $Z(N)$ models), and g_0 is an arbitrary reference conductance.

Using (1*a*) and (1*b*), the conductance G of any two-terminal array can be calculated as long as it is reducible through sequential series and parallel operations (e.g. figures 1(*b*), (*c*)). However, this is not sufficient if the array is irreducible as in figure 1(*a*). It is

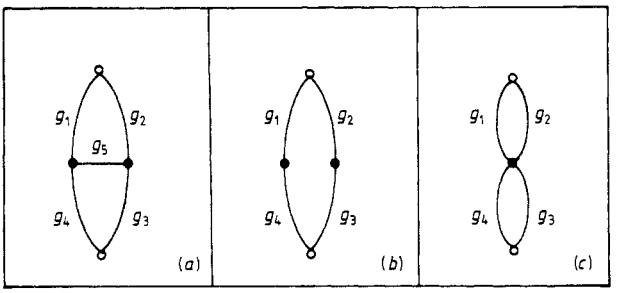

Figure 1. Two-terminal planar arrays of conductances $\{g_i\}$ (○ and ● respectively denote terminal and internal nodes): (*a*) self-dual $b = 2$ Wheatstone bridge; (*b*) and (*c*) are respectively 'broken' and 'collapsed' graphs of (*a*).

precisely this more general situation which can be solved by using the BCM. For an arbitrary two-terminal connected graph with bond conductances $\{g_i\}$, the conductance of the graph is given by $G(\{g_i\}) = N(\{g_i\})/D(\{g_i\})$ where

$$N\{g_i\} = \sum_{\substack{T \equiv \text{spanning} \\ \text{trees on } G}} \prod_{i \in T} g_i \tag{3a}$$

and

$$D\{g_i\} = \sum_{\substack{\tilde{T} \equiv \text{spanning} \\ \text{trees on } \tilde{G}}} \prod_{i \in \tilde{T}} g_i. \tag{3b}$$

The graph $\tilde{G}$ is obtained from G by adding a unit conductance between the terminals, and a spanning tree is a connected subgraph which covers all sites and contains no loops (see e.g. Wu 1982). Because of this form of N and D, they are uniquely determined once the ratio $G = N/D$ is given.

Since N and D are multilinear in the $\{g_i\}$ (Mason and Zimmerman 1960), if the jth bond of the graph is 'broken', i.e. $g_j = 0$ is imposed, the graph conductance G_j^{b} becomes,

$$G_j^{\mathrm{b}}(\{g_i\}') = N_j^{\mathrm{b}}(\{g_i\}')/D_j^{\mathrm{b}}(\{g_i\}') \tag{4}$$

where the indices j and b indicate that the jth bond has been broken, and the set $\{g_i\}'$ excludes g_j. Similarly, 'collapsing' the jth bond, i.e. imposing $g_j = \infty$, yields a new conductance G_j^{c} equal to,

$$G_j^{\mathrm{c}}(\{g_i\}') = N_j^{\mathrm{c}}(\{g_i\}')/D_j^{\mathrm{c}}(\{g_i\}'). \tag{5}$$

From the multilinearity of both N and D, we then obtain,

$$N(\{g_i\}) = N_j^{\mathrm{b}}(\{g_i\}') + g_j N_j^{\mathrm{c}}(\{g_i\}') \tag{6}$$

and

$$D(\{g_i\}) = D_j^{\mathrm{b}}(\{g_i\}') + g_j D_j^{\mathrm{c}}(\{g_i\}'). \tag{7}$$

The sequential use of these equations (together with equations (1*a*) and (1*b*)) is what we call the BCM; it greatly simplifies the calculation of conductances.

Let us illustrate the procedure on the Wheatstone bridge of figure 1(*a*). After operating on the central bond of this figure, the broken and collapsed arrays respectively indicated in figures 1(*b*) and 1(*c*) are obtained. By using equations (1*a*) and (1*b*), we find

$$G_5^{\mathrm{b}} = \frac{N_5^{\mathrm{b}}}{D_5^{\mathrm{b}}} = \frac{g_1g_2g_3 + g_1g_2g_4 + g_1g_3g_4 + g_2g_3g_4}{g_1g_2 + g_1g_3 + g_2g_4 + g_3g_4} \tag{8}$$

and

$$G_5^{\mathrm{c}} = \frac{N_5^{\mathrm{c}}}{D_5^{\mathrm{c}}} = \frac{g_1g_3 + g_2g_3 + g_1g_4 + g_2g_4}{g_1 + g_2 + g_3 + g_4}. \tag{9}$$

Therefore, by using (6) and (7),

$$G = \frac{g_1g_2g_3 + g_1g_2g_4 + g_2g_3g_4 + g_1g_3g_4 + (g_1g_3 + g_2g_3 + g_1g_4 + g_2g_4)g_5}{g_1g_2 + g_1g_3 + g_2g_4 + g_3g_4 + (g_1 + g_2 + g_3 + g_4)g_5} \tag{10}$$

which is the well-known Wheatstone bridge result. The multilinear character of the numerator and denominator of (10) is written explicitly in terms of g_5. Using the BCM, it is therefore possible to systematically reduce complicated resistor networks through simple topological operations and thereby calculate the total conductance.

As a corollary of equations (6) and (7) we obtain the following expression for the derivative

$$\partial G/\partial g_j = (N_j^{\mathrm{c}} - GD_j^{\mathrm{c}})/D. \tag{11}$$

Finally, we note that the break–collapse properties of the resistor network follow quite closely those presented in Tsallis and Levy (1981) for the q-state Potts model. This is to be expected since the resistor problem can be obtained as the $q \to 0$ limit of the Potts model (Stephen 1976, Wu 1982). To be more precise, associate with each bond a *transmissivity* (Tsallis 1981 and references therein)

$$\bar{t} = [1 - \exp(-qJ/k_{\mathrm{B}}T)]/[1 + (q-1)\exp(-qJ/k_{\mathrm{B}}T)] \tag{12}$$

where qJ is the Potts interaction parameter, and q the number of Potts states. With this

definition, parallel and series combinations of transmissivities are respectively given by

$$\tilde{t}_p = \frac{\tilde{t}_1 + \tilde{t}_2 + (q-2)\tilde{t}_1\tilde{t}_2}{1 + (q-1)\tilde{t}_1\tilde{t}_2} \tag{13}$$

and

$$\tilde{t}_s = \tilde{t}_1\tilde{t}_2. \tag{14}$$

By introducing $\tilde{t}_i = 1 - g_0/g_i$ and taking the limit $g_0/g_i \to 0$, it is straightforward to verify that for $q = 0$, the two preceding equations give the conductance of parallel and series arrays of resistors.

3. Renormalisation-group application: random-resistor network

In this section we employ the BCM for a real-space renormalisation group calculation for the random-resistor network on the square lattice. This problem has been well studied (e.g. Stinchcombe and Watson 1976, Straley 1977, Bernasconi 1978, Lobb *et al* 1981); however we propose a new averaging procedure which gives satisfactory values for the critical exponents, and the *exact* slopes for the concentration dependence of the conductivity in the pure system limit.

Consider a square lattice with the following conductance distribution for each bond

$$P(g) = (1-p)\,\delta(g - g_1) + p\,\delta(g - g_2) \qquad 0 \leqslant g_1 \leqslant g_2 \tag{15}$$

and denote the average conductivity of the lattice by $\sigma(g_1, g_2; p)$.

We shall calculate $\sigma(0, g_2; p)$ (resistor–insulator mixture) and $\sigma(g_1, \infty; p)$ (resistor–superconductor mixture) within an approximate RG framework by renormalising the self-dual $b = 2$ Wheatstone bridge (figure 1(*a*)) into a single bond. The renormalised distribution of conductances is obtained straightforwardly by using equation (10). Following along the lines of previous work (Bernasconi 1978, Mujeeb and Stinchcombe 1982 and references therein), we shall approximate this rescaled distribution by a *binary* one, namely

$$P'(g) = (1-p')\,\delta(g - g_1') + p'\,\delta(g - g_2') \tag{16}$$

where p', g_1' and g_2' are the renormalised parameters. In general, we need three RG recursion relations to calculate $\sigma(g_1, g_2; p)$, but for the particular cases of interest, namely $g_1 = 0$, $\forall g_2$ and $g_2^{-1} = 0$, $\forall g_1$, two RG recursion relations suffice. In both cases there is a δ function in the conductance distribution (at $g = 0$ for the conductor–insulator case, and at $g^{-1} = 0$ for the conductor–superconductor case) whose position remains invariant under renormalisation. Following previous work (Stinchcombe and Watson 1976, Yeomans and Stinchcombe 1978, Mujeeb and Stinchcombe 1982) we shall use the weight of this particular δ function in order to obtain the recursion relation for the occupancy probability. We have, for both the $g_1 = 0$ and $g_2^{-1} = 0$ cases,

$$p' = 2p^2 + 2p^3 - 5p^4 + 2p^5 \tag{17}$$

first obtained by Reynolds *et al* (1977) for bond percolation on the square lattice. The second recursion relation is provided by requiring that some property of the conductance distribution remains invariant under rescaling, i.e.

$$\langle f(g)\rangle_{P'} = \langle f(g)\rangle_{P_H} \tag{18}$$

where the left-hand side refers to an average on the rescaled cell at probability p' while the right-hand side refers to an average on the original 'H-shaped' cell at probability p. The choice of the function $f(g)$ appears to be arbitrary, and the cases $f(g) = g, g^{-1}$ and $\ln g$ have been employed previously (see e.g. Bernasconi 1978, Lobb *et al* 1981).

We propose the choice

$$f(g) = S(g) \equiv g/(g + g_0) \tag{19}$$

where g_0 is an arbitrary reference conductance. This is the simplest functional form that varies monotonically from 0 to 1 as g increases from 0 to ∞ while satisfying the property

$$S^{\mathrm{D}}(g) \equiv S(g^{\mathrm{D}}) = S(g_0^2/g) = 1 - S(g). \tag{20}$$

The use of such a variable obeying this probability-like transformation under duality has proved to be extremely useful in the treatment of Ising and Potts problems (Levy *et al* 1980, Tsallis 1981, Tsallis and de Magalhães 1981, de Magalhães *et al* 1982). Note also that as $g \to 0$, $S(g) \sim g/g_0$, and as $g \to \infty$, $S(g) \sim 1 - g_0/g$, so that the arithmetic and harmonic averages are recovered in these limits.

Equation (18) with $f(g) = g/(g + g_0)$ provides for the conductor–insulator problem (where we choose $g_0 = g_2$)

$$g_2' = (\tfrac{2}{3}p^2 + \tfrac{1}{2}p^3 - p^4 + \tfrac{1}{3}p^5)g_2/p' \tag{21}$$

and, for the conductor–superconductor problem (where we choose $g_0 = g_1$)

$$g_1' = \{(1-p')/[2(1-p)^2/3 + (1-p)^3/2 - (1-p)^4 + (1-p)^5/3] \ldots - 1\}g_1 \tag{22}$$

which together with equation (17) provide closed recursion relations. Notice that through the dual transformation $(p, g_2) \to (1-p, 1/g_1)$, equations (21) and (22) are the same; this property should hold for *all* self-dual Wheatstone bridges.

Analysis of the relevant linearised recursion relations near the unstable fixed point provides very satisfactory results: (i) the value $\nu = 1.43$ (Reynolds *et al* 1977); (ii) $t = s = \ln(31/17)/\ln(13/8) \simeq 1.237$, compared with the value 1.170 obtained by arithmetic and harmonic averaging and with the recent estimate of $t = 1.28 \pm 0.03$ (Derrida and Vannimenus 1982); (iii) the *exact* value of 2 for the limiting slopes of the conductivities (compared with the value 8/5 obtained from arithmetic and harmonic averaging). The full p dependences are presented in figure 2.

4. Conclusion

In conclusion, the calculation of *any* two-terminal array of conductances can be greatly simplified by performing trivial topological operations (bond 'breaking' and 'collapsing') and applying the break–collapse algorithm described herein. It avoids the use of Kirchhoff's laws and of non-linear transformations such as the star–triangle mapping. Although the break–collapse method has been exhibited on the standard Wheatstone bridge, different and larger arrays can be solved as well. The study of a few d-dimensional anisotropic random-resistor problems is presently in progress and will be published elsewhere.

In addition, we have introduced a new averaging procedure in a real-space renormalisation-group study of the random-resistor network on the square lattice. The full concentration dependence of the mean conductivity was calculated in the limiting

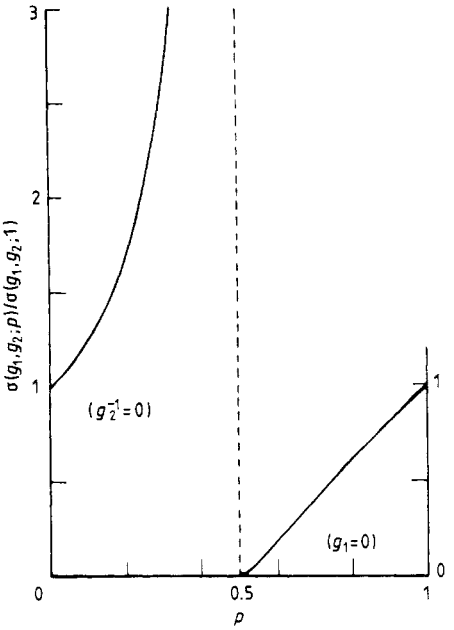

Figure 2. Concentration dependence of the square-lattice mean conductivity in both limiting cases $g_1 = 0$ (resistor–insulator mixture) and $g_2^{-1} = 0$ (resistor–superconductor mixture).

situations of a resistor–insulator ($g_1 = 0$) or a resistor–superconductor mixture ($g_2^{-1} = 0$). When mean values are taken on the variable $S(g) = g/(g + g_0)$ excellent results are obtained in spite of the small RG cluster that has been used. For the two problems, we obtain the exact critical probability $p_c = 1/2$, the *exact limiting slopes* (at $p = 0$ and $p = 1$ respectively), as well as a critical exponent $t = s \simeq 1.24$ which compares well with a quite accurate estimate of $t = 1.28 \pm 0.03$ (Derrida and Vannimenus 1982).

Acknowledgment

Two of us (CT and AC) acknowledge the hospitality offered us by the Center for Polymer Studies at Boston University where this work was initiated; CT is grateful also to the Instituto di Física Teórica/Universitá di Napoli where it was concluded. We also acknowledge useful remarks from H E Stanley; one of us (CT) is deeply indebted to R B Stinchcombe and A Mujeeb with whom he learnt important facts about the renormalisation-group treatment of random-resistor systems.

References

Alcaraz F C and Tsallis C 1982 *J. Phys. A: Math. Gen.* **15** 587
Bernasconi J 1978 *Phys. Rev.* B **18** 2185
Derrida B and Vannimenus J 1982 *J. Phys. A: Math. Gen.* **15** L557
Deutscher G 1981 *Lecture Notes in Physics* vol. 149, ed. C Castellani, C Di Castro and L Peliti (Berlin: Springer) p 26
Kirkpatrick S 1977 *Phys. Rev.* B **15** 1533

Kogut P M and Straley J P 1978 *AIP Conf. on Electrical Transport and Optical Properties of Inhomogeneous Materials* ed. D B Tanner p 382
Levy S V F, Tsallis C and Curado E M F 1980 *Phys. Rev.* B **21** 2991
Lobb C J, Frank D J and Tinkham M 1981 *Phys. Rev.* B **23** 2262
de Magalhães A C N, Schwachheim G and Tsallis C 1982 *J. Phys. C: Solid State Phys.* **15** 6791
de Magalhães A C N and Tsallis C 1981 *J. Physique* **42** 1515
Mason S J and Zimmermann H J 1960 *Electronic Circuits, Signals and Systems* (New York: Wiley) p 88
Mujeeb A and Stinchcombe R B 1982 Preprint
de Oliveira P M C and Tsallis C 1982 *J. Phys. A: Math. Gen.* **15** 2865
Reynolds P J, Klein W and Stanley H E 1977 *J. Phys. C: Solid State Phys.* **10** L167
Rosman R and Shapiro B 1977 *Phys. Rev.* B **16** 5117
Stephen M J 1976 *Phys. Lett.* **56A** 149
Stinchcombe R B and Watson B P 1976 *J. Phys. C: Solid State Phys.* **9** 3221
Straley J P 1977 *Phys. Rev.* B **15** 5733
Tsallis C 1981 *J. Phys. C: Solid State Phys.* **14** L85
Tsallis C and Levy S V F 1981 *Phys. Rev. Lett.* **47** 950
Tsallis C and de Magalhães A C N 1981 *J. Physique Lett.* **42** L227
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
Yeomans J M and Stinchcombe R B 1978 *J. Phys. C: Solid State Phys.* **11** 4095