

Physics of Finely Divided Matter

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Dynamical Processes in Random Media

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1. Introduction

A large number of percolation problems have now been thoroughly studied [1]. Most of these problems are either static in nature, such as cluster topology and structure, or of a steady-state type, such as electrical conductivity or diffusion on percolating clusters. In this paper, I discuss some aspects of dynamical percolation problems which appear to exhibit new and rich phenomena. Various portions of this work were performed in collaboration with L. de Arcangelis, A. Coniglio, H.J. Herrmann, J. Koplik and D. Wilkinson. I also thank Schlumberger-Doll Research for their support and hospitality during a leave where portions of this work were performed.

The first problem that will be treated is the breaking of random media. This process is modelled by a random fuse network in which each bond can break irreversibly when the voltage across the bond exceeds a critical value. We wish to understand how cracks form, grow and ultimately break the network. A second problem is that of dispersion in random media. This is the spreading of a dynamically neutral tracer as it is carried along in a fluid following in a porous medium.

Both problems can be formulated as random resistor network models. Breaking occurs at the most highly stressed bonds, and in the corresponding network model, these are the bonds with the highest potential drop across them. Dispersion is controlled by the bonds with the slowest flow, i.e., the bonds with the smallest potential drop across them. These insights motivate a general study of the distribution of potential drops across the bonds of a random resistor network.

2. Voltage distribution in a random resistor network

In order to discuss the distribution of potential drops analytically, we first introduce [2] a new hierarchical model to describe the percolating backbone at threshold (Fig.1). This model is a regular self-similar fractal consisting of links and blobs. Here links are defined as those bonds which, if cut, would render the network disconnected, while the remaining bonds comprise the blobs. The hierarchical model explicitly retains the self-similar links and blobs picture; an essential topological feature of the percolating backbone.

For this model, it is relatively simple to calculate a variety of geometrical quantities as a function of the iteration index, N . Now we use the fundamental relation that the number of links N_L , which equals 2^N in this model, scales exactly as $L^{1/\nu}$, where L is an effective linear dimension of the model lattice. With this connection, it is possible to obtain good numerical values for many two-dimensional percolation exponents. It is also possible to generalize these considerations to arbitrary dimensionality.

If a unit voltage is applied across the lattice, it can be readily verified for an N^{th} order hierarchy, that the possible values of the voltage drop across

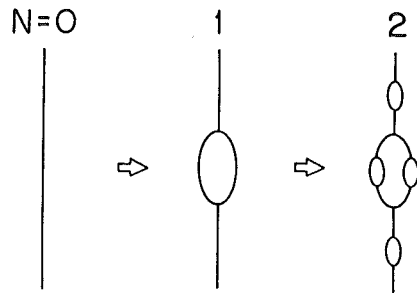


Fig.1

each bond are $V(k) \equiv 2^k/5^N$, with $k = 0, 1, 2, \dots, N$. The number of bonds with voltage drop $V(k)$ is

$$n(V(k)) = 2^N \binom{N}{k} \quad (1)$$

Since k is proportional to $\log V$, the distribution is log binomial, hence log normal in the continuum limit.

From (1) it follows immediately that $\langle V^k \rangle \equiv \sum_V n(V) V^k$ and $\langle V \rangle^k$ scale different-

ly. Thus an infinite set of exponents are required to completely specify the moments of the voltage distribution. These moments probe progressively finer details of cluster structure as k increases, ranging from the number of bonds in the backbone for $k = 0$, to the number of links for $k \rightarrow \infty$.

All of these peculiar features have been observed in our numerical simulations of the voltage distribution for a square lattice resistor network at the percolation threshold.

3. Breaking of a random fuse network

Consider a regular square lattice network in which each bond is a fuse, defined as a device which behaves as a resistor if the applied voltage V is less than a critical value V_C , and which breaks irreversibly if $V > V_C$. We treat a model where each fuse has the same value of the conductivity but a different value for the "breaking point", V_C . We shall investigate how the network breaks apart as the external voltage is increased [3].

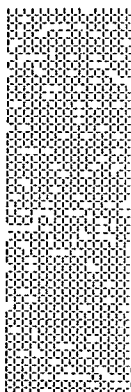
The behaviour of the model depends on the distribution of breaking points for each fuse. For simplicity, we choose a uniform distribution with average value unity and variable width w . Our detailed simulation procedure is the following: First the external voltage is set so that the breaking point of the weakest fuse is just exceeded. This weakest fuse is now broken and the new equilibrium voltages at each node of the network are then recalculated. If additional "overstressed" bonds exist, the most stressed one is now broken, and the above process is repeated. If no overstressed bonds exist, the external voltage is raised until one fuse just becomes overstressed. This bond is now broken and voltages are recalculated. This general procedure continues until the network breaks.

For $w \rightarrow 0$, the numerical simulation leads to a straight crack. This arises because the extra current that is diverted around the crack causes the bonds at the edges of the crack to become the most overstressed. Thus, a notch spontaneously grows to a straight crack which breaks the system.

For $w \rightarrow 2$, the breaking process initially resembles random percolation, as the randomness of the bond strengths is more important than the correlation effects induced by the extra voltage at the crack ends (Fig.2a). Rather suddenly, however, a new regime is reached where cracks are self sustaining and the network breaks apart (Fig.2b,c). These qualitative features can be seen more clearly by measuring the external voltage across the network and the network conductivity as a function of the number of broken bonds (Fig.3). These quantities show clearly the two general regimes of behaviour mentioned above: Independent crack initiation when relatively few bonds are broken, and spontaneous crack growth at the later stages of the breaking process.



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(b) 508 bonds



(c) 535 bonds

Fig. 2a,b,c

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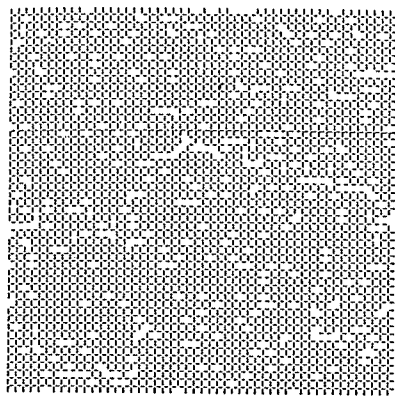
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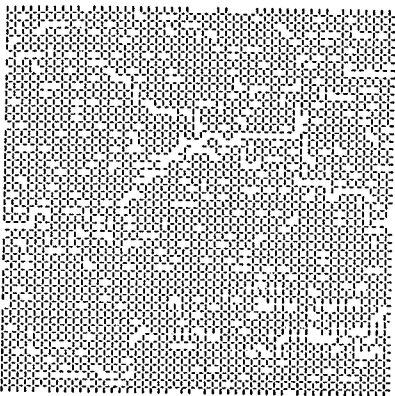
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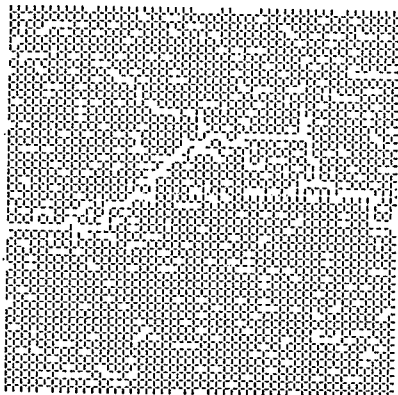
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(a) 393 bonds broken



(b) 508 bonds broken



(c) 535 bonds broken

Fig. 2a,b,c

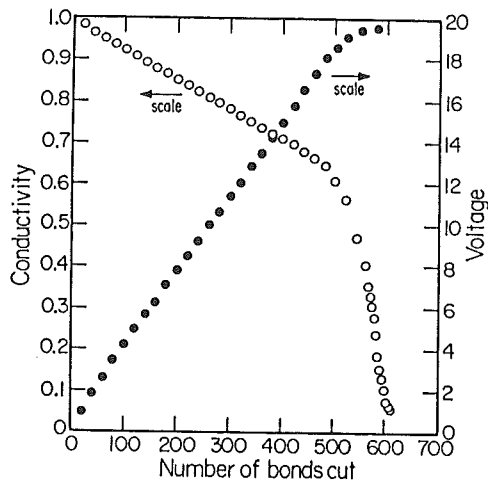


Fig. 3

At this point more theoretical interpretation is needed, and many interesting questions can be asked. For example, is the breaking process a first or a higher-order transition? How does the value of the external breaking potential and its distribution depend on the initial distribution of randomness? Is it possible to give some insight of the crack size distribution? What is the time dependence of breaking processes? These questions suggest promising directions for future research.

4. Hydrodynamic dispersion

Consider injecting a spatially localized pulse of a dynamically neutral contaminant in a fluid flowing in a porous medium. Due to the divergence and convergence of streamlines, and the different velocities of various streamlines, the pulse of tracer particles will spread as it moves downstream. This is the phenomenon of dispersion [4-6]. More quantitatively, dispersion can be quantified by $\langle r^2 \rangle - \langle r \rangle^2 = 2D_{//} t$,

where $\langle r \rangle$ is the average position of the contaminant pulse parallel to the flow, t is the time, and $D_{//}$ is defined as the dispersion

coefficient. In this section, various aspects of dispersion phenomena in random media will be discussed.

We formulate the problem in terms of a resistor network model. In a steady current flow, the mean and the mean-square transit time for a tracer to traverse the network will be

calculated. There are two basic rules for the tracer motion :

- (i) the time required for a tracer to pass through a bond which carries a current I is proportional to I^{-1} (convective flow), and
- (ii) the probability for the tracer to enter a particular bond emanating from a node equals the fractional current entering the bond (perfect mixing).

In a mean-field type approximation, the dispersion coefficient can be found by replacing the network by a linear sequence of tubes and mixing chambers, which are separated by the original lattice spacing, λ . Each tube contains all the streamlines of the network which are assumed to not interact, while perfect mixing between all streamlines occurs at the mixing chamber. In a frame of reference moving at the average flow velocity, U , a given tracer particle is sometimes moving faster and sometimes slower than U . If the distribution of individual stream velocities is not pathological, the tracer executes a random walk of step length λ and time between steps $\tau \sim \lambda/U$. This leads immediately to a dispersion coefficient $D_{//} \sim \lambda^2/\tau \sim U\lambda$.

This mean-field picture is a useful first approximation, but it is inadequate because it neglects the possibility of tracer particles getting stuck in very slow bonds. This effect can be appreciated simply in an almost balanced Wheatstone bridge (Fig.4). Suppose the bond conductivities

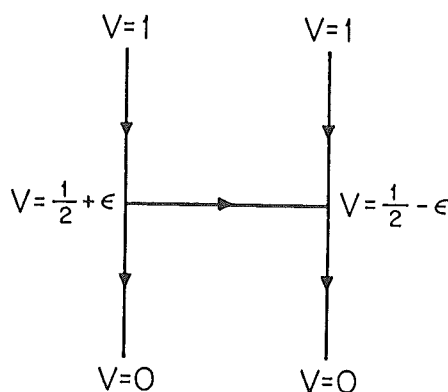


Fig.4

are such that the voltages at the midpoints of the bridge are $1/2 \pm \epsilon$. Since the current flowing in the vertical bond is approximately $1/2$, the transit time going straight down the bridge is approximately

$(1/2)^{-1} + (1/2)^{-1} = 4$. However, if the tracer takes the horizontal path (with probability proportional to ϵ), the transit time is of order $1/\epsilon$. In fact it is readily shown that $\langle t \rangle = 5 + O(\epsilon)$ and $\langle t^2 \rangle \sim 1/\epsilon$. Since $D_{//} \sim \langle t^2 \rangle - \langle t \rangle^2$, the minimum value

of ϵ must be cutoff in order to obtain a finite result. This cutoff originates in molecular diffusion; the transit time across any bond cannot be any longer than the molecular diffusion time. In a random but well-connected medium, a uniform distribution of slow bonds down to the diffusion cutoff is expected. Averaging over such a distribution leads ultimately to $D_{//} \sim U\lambda \log(U\lambda/D_{mol})$; where D_{mol} is the molecular diffusion coefficient.

Thus the effects of molecular diffusion are quite important and they are required to give a sensible description of dispersion.

Turning now to dispersion on poorly connected networks, we first introduce a modified hierarchical lattice. It is obtained from figure 1 by taking the conductivity of a bond on the right edge of each bubble to be one-half the conductivity of the corresponding bond on the left edge. For this network it is possible, but somewhat tedious, to calculate the asymptotic form of the transit time distribution. A primary result of this calculation is that the dispersion coefficient is scale dependent and it may be written as $D_{//} \sim UL$, where L is the effective linear dimension of the lattice as defined in section 2. As a result of this scale dependence, a localized pulse of tracer will spread out at a rate faster than $t^{1/2}$. This anomalously fast dispersion is similar, but also complementary, to the phenomenon of anomalously slow diffusion on self-similar structures [7].

Attempts to account for the effects of molecular diffusion are in progress. An important new feature for poorly connected media is that regions of stagnation can

form, rather than this section alone, and on types of stagnation on the percolation vary as $U^2 \xi^2 /$ (dependent) molecular understanding of diffusive and diffusive

form, rather than isolated slow bonds, for which the discussion of the beginning of this section applied. Furthermore, stagnation can occur in large blobs in the backbone, and on dead ends; blobs and dead ends play the same role in dispersion. Both types of stagnation need to be included to give a full description of dispersion on the percolating cluster. For this case, it has been suggested [8] that $D_{//}$ will vary as $U^2 \xi^2 / D_{ant}$, where ξ is the correlation length, and D_{ant} is the (scale-dependent) molecular diffusion coefficient on the percolating cluster. The complete understanding of this result, and the crossovers caused by the interplay of convective and diffusive effects, should be fruitful areas for new results.

- (1) See e.g. D Stauffer Phys Rep 54, 1
- (2) de Arcangelis et al Phys Rev B 31, 4725 (1985)
- (3) de Arcangelis et al J Phys. Lett 46, L585 (1985)
- (4) P. A Saffman J. Fluid Mech 6 321 (1959); 7, 144 (1960)
- (5) M. Sahimi et al Chem Engr. Comm. 23, 329 (1983); Chem Engr. Sci to appear
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- (7) Y. Gefen et al Phys Rev Lett 50, 77 (1982)
- (8) P. G. de Gennes J Fluid Mech 136 189 (1983)