

Clogging Time of a Filter

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We study the time until a filter becomes clogged due to the trapping of suspended particles as they pass through a porous medium. This trapping progressively impedes and eventually stops the flow of the carrier fluid. We develop a simple description for the pore geometry and the motion of the suspended particles which, together with extreme-value statistics, predicts that the distribution of times until a filter clogs has a power-law long-time tail, with an infinite mean clogging time. These results and its consequences are in accord with simulations on a square lattice porous network.

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In this letter, we investigate the time required for a filter to clog. In a typical filtration process, a dirty fluid is “cleaned” by passing it through a porous medium to remove the suspended particles. The medium enhances filtering efficiency by increasing both the available filter surface area for trapping suspended particles, as well as the exposure time of the suspension to the active surfaces. Such a mechanism is the basis of water purification, air filtration, and many other separation processes [1,2]. As suspended particles become trapped, the fluid permeability of the medium gradually decreases, and eventually the filter becomes clogged. Determining the time dependence of this clogging is basic to predicting when a filter is no longer useful, either because of reduced throughput or reduced filtration efficiency, and should be discarded.

We develop a minimalist model which provides an intuitive understanding for the clogging of a filter. Our analytical results are based on representing the clogging of a porous medium by the clogging of a single parallel array of pores which are blocked in decreasing size order (Fig. 1). This geometrical reduction is based on previous filtration studies which showed that, when particles and pores are of comparable size, clogging preferentially occurs in the upstream end of the network [3,4]. A single parallel array of pores represents the ultimate limit of this gradient-controlled process.

The approximation of size-ordered blocking is based on the commonly-used picture that a particle has a probability proportional to the relative flux to enter a given unblocked pore from among the outgoing pores at a junction [2,3,5]. For Poiseuille flow, this flux is proportional to r^4 , where r is the pore radius. Size-ordered blocking arises if the exponent 4 in this flow-induced pore entrance probability is replaced by ∞ . For a broad distribution of pore radii, this is not a drastic approximation. This ordering also provides a direct correspondence between the radius of the currently-blocked pore and the time until it is blocked. We then use extreme value statistics [6] to compute the radius distribution of the last few unblocked pores, together with the connection between the radius of the currently blocked pore and the network permeability, to determine the distribution of filter clogging times. The predictions of this simple modelling are in

good agreement with Monte Carlo simulations of clogging on a square lattice porous network.

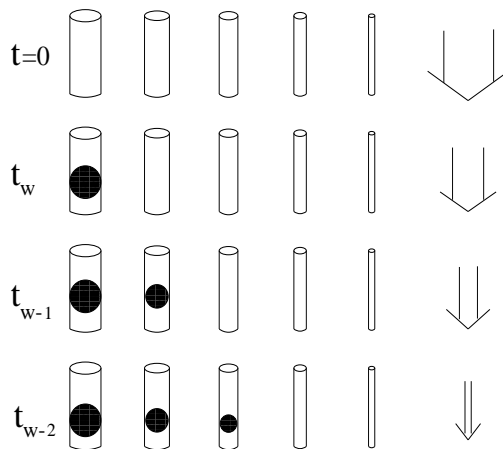


FIG. 1. Illustration of the clogging evolution in a parallel array of w bonds of distributed radii. Bonds are blocked in decreasing size order. The overall flow decreases more slowly in the latter stages (vertical arrow). The total flow is inversely related to the time until each blockage event, t_k , with $k = w, w - 1, \dots$.

In our simulations of clogging, we consider a lattice porous medium whose bonds represent pores. We assume Poiseuille flow, in which the fluid flux through a bond of radius r_i is proportional to $-r_i^4 \nabla p$, where ∇p is the local pressure gradient when a fixed overall pressure drop is imposed. Dynamically neutral suspended particles traverse the medium in accordance with the local flow and the flow-induced entrance probability at each junction. A crucial feature is that particles are injected at a *finite* rate which is proportional to the overall fluid flux, corresponding to a fixed non-zero density of suspended particles in the fluid [5]. This is in distinction to many previous studies of filtration, where particles were injected singly and tracked until trapping occurred [2–4]; this corresponds to an arbitrarily dilute suspension. While useful for understanding the percolation process induced by clogging [3–5,7], the time evolution of filtration necessitates the consideration of a finite-density suspension.

For the trapping mechanism we adopt size exclusion, in which a particle of radius r_{particle} is trapped within the first bond encountered with $r_{\text{bond}} < r_{\text{particle}}$ [5]. We assume that a trapped particle blocks a pore completely and permanently. The difference between partial and total blockage in a single trapping event appears to be immaterial for long-time properties [4]. The overall trapping rate is then controlled by the relative sizes of particles and bonds. For simplicity and because it occurs in many types of porous media [8], we consider the Hertz distribution of pore and particle radii, respectively,

$$b(r) = 2\alpha r e^{-\alpha r^2}, \quad \text{and} \quad p(r) = 2r e^{-r^2}, \quad (1)$$

with the ratio between the average bond and particle radii $s = 1/\sqrt{\alpha}$ a basic parameter which determines the nature of the clogging process.

First consider the case where pores are larger than particles ($s \gtrsim 1$). Many particles must be injected before a sufficiently large particle arises which can block the largest pore. Since the permeability of the system remains nearly constant when only a few pores are blocked, the time between successive particle injection events during this initial stage is nearly constant. Once the largest pores are blocked, it takes many fewer particles to block the remaining smaller pores and clogging proceeds more quickly. Thus the clogging time is dominated by the initial blockage events.

To estimate the clogging time in this large-pore limit, let us find the number of injection events before encountering a particle large enough to block the largest bond. The radius of this bond is given by the criterion [6]

$$\int_{r_{\text{max}}}^{\infty} 2\alpha r e^{-\alpha r^2} dr = \frac{1}{w}, \quad (2)$$

that one bond out of w has radius r_{max} or larger. This gives $r_{\text{max}}^{(b)} = s\sqrt{\ln w}$. Similarly, the largest particle out of N has radius $r_{\text{max}}^{(p)} = \sqrt{\ln N}$. Thus $N_1 \approx w^{s^2}$ particles typically need to be injected before one occurs which is large enough to block the largest bond. Continuing this picture sequentially, the radius of the k^{th} -largest bond is given by Eq. (2), but with $1/w$ replaced by k/w ; this gives $r_k = s\sqrt{\ln(w/k)}$. Therefore $N_k \approx (w/k)^{s^2}$ particles need to be injected before the k^{th} -largest bond is blocked. Since N_k decreases rapidly with k , the initial blockage events control the clogging time T , whose lower bound is given by $T \approx N_1/w > w^{s^2-1}$. The factor of $1/w$ arises because the particle injection rate is proportional to w for a fixed-density suspension and a fixed pressure drop. However, there is considerable particle penetration in the large-pore limit, so that the equivalence to the clogging of a parallel bond array is not really appropriate and the bound on T is quite crude.

The latter case where pores are smaller than particles ($s \lesssim 1$) is simpler, as each particle injection event typically leads to the clogging of the first pore entered. Now the initial pores are blocked quickly, while later pores

are blocked more slowly because the overall flow rate decreases significantly near the end of the clogging process. We argue that clogging is dominated by the time of these later blockage events. First, let us determine how the flow rate varies as the last few bonds get blocked. Since only the smallest bonds remain open near clogging, the permeability is determined by these smallest radii. We estimate the radius of the k^{th} smallest bond from

$$\int_0^{r_k} 2\alpha r e^{-\alpha r^2} dr = \frac{k}{w}, \quad (3)$$

which gives $r_k = s\sqrt{k/w}$. The permeability of a parallel bundle of these k smallest pores is then

$$\kappa(k) = \sum_{j=1}^k r_j^4 \approx s^4 \sum_{j=1}^k (j/w)^2 \sim s^4 k^3/w^2, \quad (4)$$

while the initial system permeability (obtained by setting $k = w$ above) is simply $\kappa(w) = s^4 w$.

Since the overall fluid flow is proportional to the permeability for a fixed pressure drop, the time increment t_k between blocking the $(k-1)^{\text{st}}$ -smallest and k^{th} -smallest pore scales as

$$t_k = \frac{\kappa(w)}{w\kappa(k)} \sim \frac{w^2}{k^3}. \quad (5)$$

Here the factor w in the denominator again accounts for a particle injection rate proportional to w . The clogging time T is now dominated by the time to block the smallest bond, so that $T > t_1 \propto w^2$.

Thus the relative pore and particle radii drives a transition in the clogging process. For $s^2 > 3$, clogging is dominated by the initial blockage events and $T > w^{s^2-1}$, while for $s^2 < 3$, the last events dominate, leading to $T > w^2$. For the small-pore limit, we may carry the analysis further and obtain the distribution of clogging times. This distribution gives a *divergent* mean clogging time; nevertheless, suitably defined moments of the clogging time distribution scale as w^2 .

We find the clogging time distribution in terms of the radius distribution of the smallest bond, since this bond ultimately controls clogging in a single parallel bond array. For the Hertz distribution, the probability that a given bond has a radius greater than or equal to r , $B_{>}(r)$, is

$$B_{>}(r) = \int_r^{\infty} 2\alpha r e^{-\alpha r^2} dr = e^{-\alpha r^2}. \quad (6)$$

Then the radius distribution of the smallest bond from among w , $S_w(r)$, is given by [6]

$$S_w(r) = w b(r) B_{>}(r)^{w-1} = 2\alpha w r e^{-\alpha w r^2}. \quad (7)$$

The first equality expresses the fact that one of the w bonds has (smallest) radius r , with probability $b(r)$, and the other $w-1$ must have radii larger than r .

From the basic connections between permeability, pore radius, and time scale (Eqs. (4) and (5)), and the fact that the clogging time is dominated by t_1 , we deduce

$$T \sim t_1 \approx \frac{\kappa(w)}{w\kappa(1)} = \frac{s^4}{r_1^4}, \quad (8)$$

while the clogging time distribution, $P_w(T)$, is directly related to the smallest bond radius distribution through $P_w(T) dT = S_w(r) dr$. Using Eqs. (7) and (8), we obtain the basic result (independent of system length)

$$P_w(T) \sim \frac{w}{T^{3/2}} e^{-w/T^{1/2}}. \quad (9)$$

The power law should apply in the time range $w^2 < t < w^2 N^2$. The former corresponds to the size of the typical smallest pore in a single realization, s/\sqrt{w} , while the latter corresponds to the smallest pore from among N realizations of the system, s/\sqrt{Nw} . The essentially singular short-time cutoff arises from those realizations where the smallest bond happens to be anomalously large. Other coincident particle and bond radius distributions lead to qualitatively similar forms for $P_w(T)$, but with a different exponent value. For example, if $S_w(r) \sim r^\mu$ as $r \rightarrow 0$, then the long-time exponent in $P_w(T)$ is $-(\mu + 5)/4$.

We test our predictions by Monte Carlo simulations of the motion and trapping of suspended particles in a lattice porous medium. Each bond (with unit length) corresponds to a pore and the sites represent pore junctions. The algorithm is event-driven to bypass the physical slowing down of the flow as clogging is approached. At an intermediate stage, there are a finite number of particles in the network, consistent with a unit pressure gradient and a constant-density suspension. Particles are defined to be always at lattice sites and each (including newly injected particles) evolves by either: (a) moving to the next “downstream” site, (b) blocking the downstream bond it has entered, or (c) remaining stationary. The probability of each of these possibilities is defined so that the process corresponds to particles moving, on average, at the local velocity.

To achieve this, each particle at a junction selects an outgoing bond i , with entrance probability equal to the fractional flux into this bond. In an update, a particle attempts to move to the next junction, via the pre-selected bond, with probability proportional to v_i/v_{\max} , where v_i is the local bond velocity and v_{\max} is the instantaneous largest particle velocity in the system. If the attempt occurs, the particle either moves to the next downstream site, or, if the particle is too large, blocks the bond. After a single update of all particles, the time is incremented by $\Delta t = 1/v_{\max}$ to ensure that, on average, a particle moving along bond i has speed v_i .

After all particles undergo their move attempts, $\mathcal{N} \propto \phi \Delta t$ new particles are injected at the upstream end of the network, where ϕ is the overall fluid flux. This ensures the correct absolute velocity for each particle. Since the time increment systematically increases as bonds get

blocked, the particle injection rate progressively slows. Our approach is distinct from earlier multiparticle simulations of filtration, where the injection rate was decoupled from the overall flow [5].

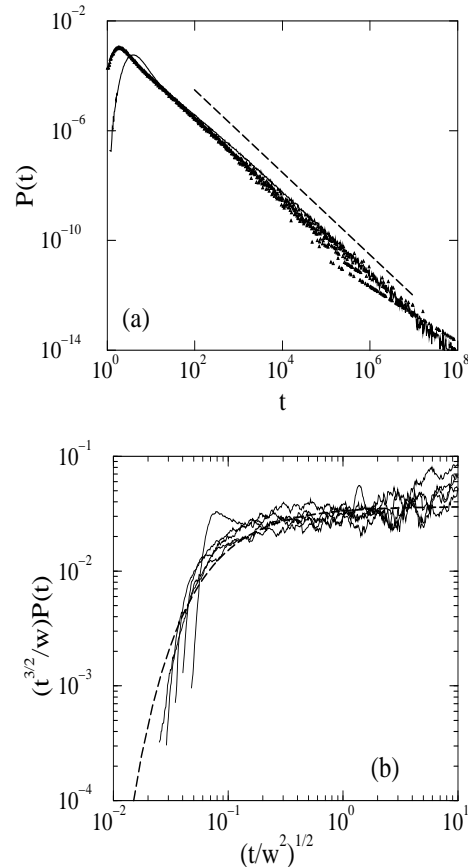


FIG. 2. (a) Distribution of clogging times on the bubble model (continuous curve) and on a square lattice (data points) for system width 20 and length 10. Time units correspond to an initial injection rate of 0.05 particles per unit width per unit time. The straight line has slope $-3/2$. (b) The scaled clogging time distribution for square lattices of length 10 and widths 20, 40, 60, 100, and 200 (right to left). The data have been smoothed over a 12-point range. The heavy dashed line is the prediction of Eq. (9).

In Fig. 2(a), we plot the clogging time distribution from simulations on the square lattice and the bubble model of the same spatial extent. The latter is a series of parallel bond arrays with perfect mixing at each junction; this idealized system accounts for geometrical aspects of clogging [4]. The agreement between the two distributions is remarkably good, suggesting that the schematic evolution proposed in Fig. 1 is quantitatively correct. The tails of the two distributions in Fig. 9(a) are well-fit by a power law with exponent $-3/2$. As a further test of Eq. (9), we plot, in scaled units, data for the clogging time distribution for systems of various widths w (Fig. 2(b)) and compare to the functional form in Eq. (9).

An important consequence of the power-law tail in the clogging time distribution is a transition in the corresponding moments. These are

$$\langle t^k \rangle = \int_0^\infty t P_w(t) dt \approx \int_{w^2}^{N^2 w^2} w t^{k-3/2} dt, \quad (10)$$

where the approximation replaces the short-time cutoff at $t \approx w^2$ in Eq. (9) by the lower limit in the integral. Consequently

$$M_k(w) \equiv \langle t^k \rangle^{1/k} \sim \begin{cases} w^2 N^{2-1/k} & k > 1/2; \\ w^2 (\ln N)^2 & k = 1/2; \\ w^2 & k < 1/2. \end{cases} \quad (11)$$

Thus the mean clogging time diverges when the number of realizations is infinite, leading to large sample to sample fluctuations in the clogging time, while the moments M_k with $k < 1/2$ should be well behaved. The transitions in the moments is illustrated in Fig. (3), where M_k behaves erratically for $k = 1$ and $1/2$, but then appears to grow as w^2 for $k \leq 1/3$.

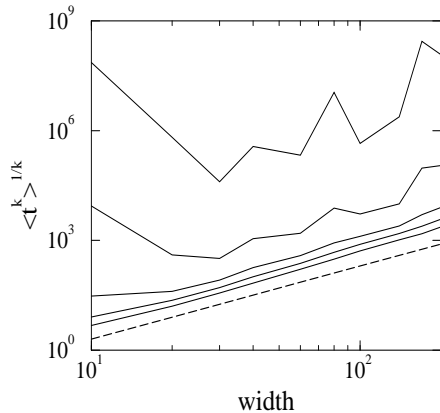


FIG. 3. Width dependence of the moments $\langle t^k \rangle^{1/k}$ for $k = 1, 1/2, 1/3, 1/4$, and $1/6$ (top to bottom). The dashed line corresponds to a w^2 dependence.

The behavior of M_k raises the question of when is a filter no longer useful. Waiting until complete clogging is impractical because of large fluctuations in the clogging time and eventual poor filter performance. It would be more useful to operate a filter only until the permeability decays to a (small) fraction of its initial value, such that reasonable flow and trapping efficiency are maintained, while minimizing fluctuations in this threshold time. We provide some preliminary results about these questions. Simulations indicate that the time to reach permeability fraction f , $t(f)$, is proportional to $w^{2/3} f^{-1/2}$ for $10^{-10} \lesssim f \lesssim 10^{-3}$, with the distribution of these fractional times progressively broadening for decreasing f (Fig. 4). Thus waiting until the permeability decays to a fixed fraction can provide at least a reliable criterion for when a filter should be discarded. A consequence of $t(f) \propto w^{2/3} f^{-1/2}$ is that the permeability decays as

$1/t^2$ over a wide range. Because of this rapid decrease, a practical filter should also have pores typically larger than particles to have a reasonable lifetime. These issues and understanding the efficiency of a filter throughout its useful life are currently under investigation.

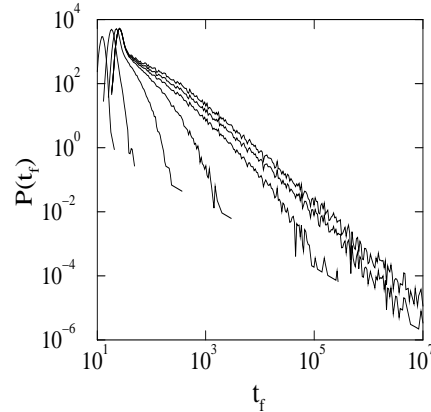


FIG. 4. Probability that a filter with $w = 200$ reaches permeability fraction f at time t_f , for $f = 4^{-n}$, with $n = 2, 4, 6, 8, 12, 16$, together with the time distribution until complete clogging (left to right).

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