

# A Billiard-Theoretic Approach to Elementary 1d Elastic Collisions

S. Redner<sup>1,\*</sup>

<sup>1</sup>*Center for BioDynamics, Center for Polymer Studies,  
and Department of Physics, Boston University, Boston, MA, 02215*

A simple relation is developed between elastic collisions of freely-moving point particles in one dimension and a corresponding billiard system. For two particles with masses  $m_1$  and  $m_2$  on the half-line  $x > 0$  that approach an elastic barrier at  $x = 0$ , the corresponding billiard system is an infinite wedge. The collision history of the two particles can be easily inferred from the corresponding billiard trajectory. This connection nicely explains the classic demonstrations of the “dime on the superball” and the “baseball on the basketball” that are a staple in elementary physics courses. It is also shown that three elastic particles on an infinite line and three particles on a finite ring correspond, respectively, to the motion of a billiard ball in an infinite wedge and on a triangular billiard table. It is shown how to determine the angles of these two sets in terms of the particle masses.

PACS numbers: 01.40.-d, 45.05.+x, 45.50.-j

## I. INTRODUCTION

A standard discussion topic in freshman mechanics courses is elastic collisions. There are two very nice lecture demonstrations to accompany this topic [1]. The first is the “dime on the superball”. Here one carefully places a dime on top of a superball and drops this composite system on a hard floor. Before doing so, the instructor asks the class to guess the maximum height of the dime  $h_{\max}$  compared to the initial height  $h_0$ . For perfectly elastic collisions and in the limit where the mass of the dime is vanishingly small, it is easy to show that  $h_{\max} = 9h_0$  [2]! While this theoretical limit cannot be achieved, the dime can easily hit the ceiling of a normal classroom when the superball is dropped from chest level.

Another demonstration of this genre is to carefully place a baseball (hardball) of mass  $m$  on top of a basketball of mass  $M$  and then drop the two together. Now the question for the class is: how high does the basketball rise after collision with the floor? It turns out that for  $m = M/3$  (which is close to the actual mass ratio for a baseball and a basketball) and again for perfectly elastic collisions, the basketball hits floor and stays there! However, warn the class to beware of the rapidly moving baseball! In the theoretically ideal situation, it rises to a height  $h_{\max} = 4h_0$ . Working out these two examples is left as exercises for the reader.

The extension of this simple two-particle system to arbitrary mass ratios presents many interesting and unexpected challenges. In what follows, we generally ignore gravity because it plays a negligible role during the collisions. If the upper mass is much larger than the lower mass, then for two separated masses that approach an elastic barrier, there will be a large number of collisions before the two masses diverge and ultimately recede from the barrier. This dynamics can, in principle, be analyzed

by applying momentum conservation to map out the particle trajectories. This approach is tedious, however, and does not provide physical insight (see Appendix).

The goal of this article is to present a simple connection between the motion of few-particle elastically colliding systems in one dimension and a corresponding billiard system. For two particles and an elastic barrier the corresponding billiard ball moves in a two-dimensional wedge-shaped billiard table with elastic and specular reflection each time the ball hits the boundary of the table [3, 4, 5, 6]. Specular means that the angle of incidence equals the angle of reflection. This description can be greatly simplified by recognizing that specular reflection at a boundary is geometrically identical to passing straight through the boundary, where on the other side of the boundary there is an identical image of the wedge. By repeating this construction, the end result is that billiard motion in the wedge is equivalent to a straight trajectory in a plane that is “tiled” by a fan of wedges. By this equivalence, it is easy to completely solve the collision history of the original two-particle and barrier system.

We then extend this approach to treat three elastically-colliding particles of arbitrary masses on an infinite one-dimensional line. Here we ask the question: how many collisions occur when two cannonballs are approaching, with an intervening elastic ping-pong ball that is rattling between them [7]? This system can again be mapped onto the motion of a billiard ball in an infinite wedge whose opening angle depends on the three masses. Finally, we discuss the motion of three particles on a finite ring [8, 9]. This system can be mapped onto the motion of a billiard ball on a triangular table. Through this connection, we can gain many useful insights about the collisional properties of the three particle on the ring.

These exactly soluble few-body systems naturally open new issues. For example, what happens when the number of particles becomes large? In one dimension, the momentum distribution of a polydisperse system of elastic particles converges to a finite- $N$  version of the Gaussian distribution [10]. However, transport properties appear

---

\*Electronic address: redner@bu.edu

to be anomalous. In particular, there is a lingering controversy about the nature of heat flow through such a system [11] and the nature of the thermodynamic limit of this system is not yet fully understood.

On a broader scope, one may naturally inquire about the roles of inelastic collisions and the spatial dimension on the dynamics. This is a natural entry to the burgeoning field of granular media [12]. While this area is beyond the scope of this work, it is worth mentioning a few relevant topics. A particularly intriguing feature of inelastic systems is the phenomenon of “inelastic collapse”, where clumps of particles with negligible relative motion form. This collapse occurs when the number of particles is sufficiently large or when collisions are sufficiently inelastic [13, 14, 15]. Some of the methods described here may be useful to understand these systems. The inclusion of other natural parameters also leads to a wealth of new effects. For example, a variety of collisional transitions occur when inelastic particles are pushed by a massive wall [16], while the presence of gravity in an elastic system of two particles and a wall leads to both quasi-periodic and chaotic behavior [17]. Again, a billiard-theoretic perspective may provide helpful insights into these systems.

In the next section, we discuss the dime on the superball and the baseball on the basketball by elementary means. Then in Sec. III, we show how to map these systems onto the motion of a billiard ball in a wedge domain. This same approach is used to show the equivalence of three particles on an infinite line to a billiard ball in an infinite wedge in Sec. IV, and the equivalence of three particles on a ring to a triangular billiard in Sec. V. A brief discussion is given in Sec. VI.

## II. DIME ON SUPERBALL & BASEBALL ON BASKETBALL

To aid in the analysis of the dime on the superball, it is helpful to imagine that the two particles are separated. Again, gravity is neglected throughout the collisions; its only role is to give the final height of the dime in terms of its velocity immediately after the last collision. Fig. 1 shows the velocities of the dime and superball under the assumption that the mass  $m$  of the dime is negligible compared to that of the superball ( $M$ ), and that all collisions are perfectly elastic. The following collision sequence occurs:

- (i) The dime and the superball both approach the ground with velocity  $-v$ .
- (ii) The superball hits the ground and reverses direction so that its velocity is  $+v$ .
- (iii) For  $m/M \rightarrow 0$ , the center-of-mass coincides with the center of the superball. In this reference frame, the dime approaches the superball with velocity  $-2v$ .

- (iv) After the dime-superball collision in the center-of-mass frame, the dime moves with velocity  $+2v$ , while the superball remains at rest.
- (v) Returning to the original lab frame, the superball moves with velocity  $+v$ , while the dime moves at velocity  $+3v$ . This velocity of  $+3v$  leads to the dime rising to a final height that is nine times that of the superball in the presence of gravity.

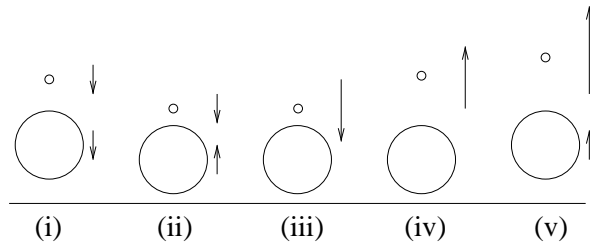


FIG. 1: Collision sequence for the dime-superball system. The arrows (drawn to scale) denote the velocities at each collision stage.

For the baseball on top of the basketball, the same analysis now gives the following collision sequence (again assuming perfectly elastic collisions):

- (i)-(ii) The basketball hits the ground with velocity  $-v$  and reverses direction so that its velocity is  $+v$ .
- (iii) For  $m/M = 1/3$ , the center-of-mass has velocity  $+v/2$ . In the center-of-mass reference frame, the baseball has velocity  $-3v/2$ , while the basketball has velocity  $+v/2$ .
- (iv) After the collision between the two balls in the center-of-mass frame, their velocities are reversed.
- (v) In the original lab frame, the baseball has velocity  $+2v$ , while the basketball is at rest.

In both these cases, there are just two collisions – an initial collision of the lower ball with the floor and a second collision between the two balls. Subsequently, the upper ball moves faster than the lower ball and there would be no more collisions in the absence of gravity. However, if the upper ball is heavier than the lower ball, then there will be many collisions before the two balls recede from the floor and from each other. How many collisions occur in total for this system? What are the details of the collision sequence? These questions should be simple to answer, since only energy and momentum conservation are involved. However, when the upper ball is much heavier than the lower ball the number of collisions is large and a direct solution is tedious. As we discuss in the next section, there is an elegant mapping of this collision problem to an equivalent billiard system that provides an extraordinary simplification.

### III. BILLIARD MAPPING

We now map the problem of two colliding particles and an elastic barrier into an equivalent billiard system. From this approach, the entire particle collision history can be inferred in a simple geometric manner. To be general, suppose now that the particles have masses  $m_1$  and  $m_2$  and are located, respectively, at  $x_1$  and  $x_2$ , with  $x_1 < x_2$  (and  $x_1, x_2 > 0$ ). The trajectories of the two particles in one dimension are equivalent to the trajectory  $(x_1(t), x_2(t))$  of an effective billiard ball in the two-dimensional domain defined by  $x_1, x_2 > 0$  and  $x_1 < x_2$ . The billiard ball hitting the boundary  $x_1 = 0$  corresponds to a collision between the lower particle and the floor, while the ball hitting the boundary  $x_1 = x_2$  corresponds to a collision between the two particles.

Now define the following ‘‘billiard rescaling’’ [3, 4, 5, 6]:

$$y_i = x_i \sqrt{m_i} \quad w_i = v_i \sqrt{m_i},$$

for  $i = 1, 2$ . In these coordinates, the constraint  $x_1 < x_2$  becomes

$$y_2 > \sqrt{\frac{m_2}{m_1}} y_1.$$

Thus the allowed region is now a wedge-shaped domain (see Fig. 2) with opening angle

$$\alpha \equiv \tan^{-1} \sqrt{\frac{m_1}{m_2}}. \quad (1)$$

The crucial feature of this rescaling is that it ensures that all collisions of the billiard ball with boundary of the domain are specular. To demonstrate this point, we take the energy and momentum conservation statements,

$$\begin{aligned} \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 &= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \\ m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2', \end{aligned} \quad (2)$$

where the prime denotes a particle velocity after a collision, and rewrite these conservation laws in rescaled coordinates to give

$$\begin{aligned} w_1^2 + w_2^2 &= w_1'^2 + w_2'^2 \\ \sqrt{m_1} w_1 + \sqrt{m_2} w_2 &= \sqrt{m_1} w_1' + \sqrt{m_2} w_2'. \end{aligned} \quad (3)$$

The first of these equations states that the speed of the billiard ball is unchanged by a collision. The second equation can be rewritten as  $(\sqrt{m_1}, \sqrt{m_2}) \cdot (w_1, w_2)$  remains constant in a collision. Since the vector  $(\sqrt{m_1}, \sqrt{m_2})$  is tangent to the constraint line  $y_2 = y_1 \sqrt{m_2/m_1}$ , the projection of the rescaled velocity onto this line is constant in a particle-particle collision. It is also intuitively clear that in a particle-wall collision the rescaled velocity is also preserved. As a result, the collision sequence of two elastically-colliding particles and an elastic barrier in one dimension is completely equivalent to the trajectory of a

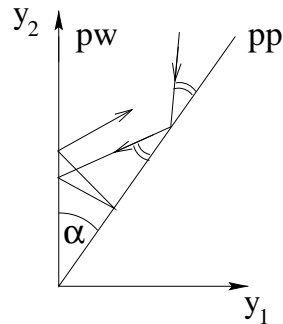


FIG. 2: Allowed wedge in  $y_i$  coordinates. A sample billiard ball trajectory is shown. Hitting the  $y_2$  axis corresponds to a particle-wall collision (denoted by  $pw$ ), while hitting the line  $y_2 = y_1 \sqrt{m_2/m_1}$  corresponds to a particle-particle collision (denoted by  $pp$ ).

billiard ball in a two-dimensional wedge of opening angle  $\alpha$  in which each collision with the boundary is specular.

A more dramatic simplification arises in the  $y_i$  coordinates by recognizing that since each reflection is specular, the trajectory in the wedge is the same as a straight trajectory in the periodic extension of the wedge (Fig. 3). Each collision is alternately a particle-particle or a particle-wall, so that the identity of each barrier alternates between  $pp$  and  $pw$ . From this description, we immediately deduce that the collision sequence of the two-particle system ends when the trajectory of the billiard ball no longer crosses any wedge boundary. As shown in Fig. 3, when the original trajectory is extended in the manner, it will ultimately pass through six wedges. Thus five collisions (particle-wall and particle-particle) occur in total.

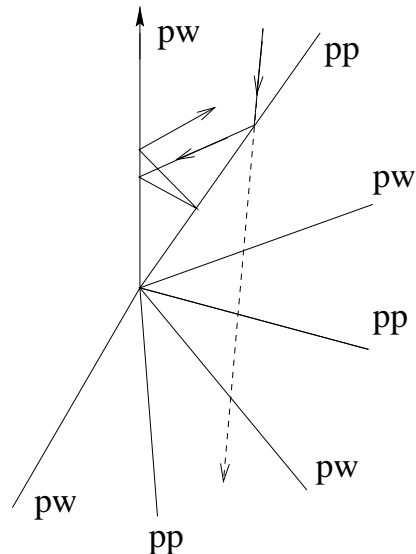


FIG. 3: Periodic extension of the allowed wedge. The trajectory in the original wedge is equivalent to the straight trajectory shown (dashed).

The maximum number of wedges that can be packed in the half plane is  $\pi/\alpha$ . A straight trajectory of the billiard ball typically passes through all these wedges. This is therefore the maximum number of collisions  $N_{\max}$  possible in the two-particle system. In the limit where  $m_1 \ll m_2$ , we thereby find (using Eq. (1))

$$N_{\max} \simeq \pi \sqrt{\frac{m_2}{m_1}}. \quad (4)$$

Thus the total number of collisions in the original particle system emerges from extremely simple geometric considerations of the equivalent billiard.

From Fig. 3, the incidence angle of the billiard ball at each boundary increases by a factor  $\alpha$  after each collision. Furthermore, using the constancy of the rescaled velocity  $w$ , one can also deduce the particle velocities at every collision stage. We now illustrate this approach by reconsidering our initial examples from this billiard-theoretic perspective.

#### IV. DIME ON SUPERBALL & BASEBALL ON BASKETBALL: A SECOND LOOK

For the dime on the superball, the opening angle of the wedge has the limiting behavior  $\alpha = \frac{\pi}{2} - \delta$ , with  $\delta \approx \sqrt{m_2/m_1}$  as  $m_2/m_1 \rightarrow 0$ . In Fig. 4, the trajectory of the corresponding billiard ball is shown in the  $y_1$ - $y_2$  coordinate system. Because the dime and the superball have the same initial velocities, the incoming trajectory in the wedge is parallel to the initial  $pp$  boundary. The distance to the  $pp$  boundary is proportional to the initial separation of the dime and the superball. (If, initially,  $x_2 = x_1 + \epsilon$ , then  $y_2 = y_1 \sqrt{m_2/m_1} + \sqrt{m_2} \epsilon$ .)

After the superball collides with the wall, the billiard trajectory is incident on the  $pp$  boundary with inclination angle  $2\delta$  (Fig. 4). After specular reflection from this boundary, the final outgoing trajectory is then inclined at an angle  $3\delta$  with respect to the horizontal. This inclination angle means that the final velocity of the dime is three times that of the superball. Thus in the presence of gravity, an ideal dime will rise to nine times its initial height.

This same result can be obtained even more simply by drawing a straight trajectory through the periodic extension of the wedges. In this case, the final trajectory is inclined at an angle of  $\frac{\pi}{2} - 3\delta$  with respect to the last periodically extended  $pw$  boundary. This construction again implies that the outgoing trajectory is inclined at an angle of  $3\delta$  with respect to the initial  $pp$  boundary.

For a basketball of mass  $m_1 = 3m$  and a baseball of mass  $m_2 = m$ , the opening angle of the wedge is now  $\alpha = 60^\circ$  (Fig. 5). Again, there are two collisions in total and by simple geometry it easily follows that the final outgoing billiard trajectory is vertical, *i.e.*,  $v'_1 = 0$  and  $v'_2 > 0$ . We can obtain the final speed  $v'_2$  by exploiting the constancy of the rescaled speed. Initially,  $\sqrt{w_1^2 + w_2^2} =$

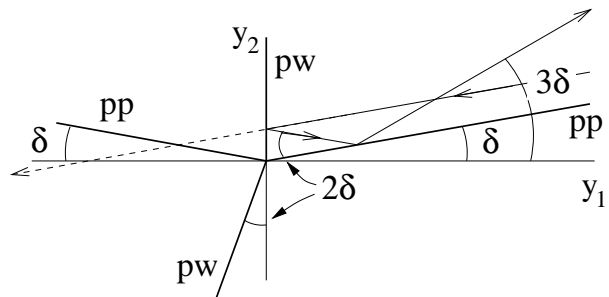


FIG. 4: Allowed wedge in  $y_i$  coordinates for the dime and superball. A trajectory corresponding to the dime and the superball approaching the wall at the same velocity is shown. The dashed line shows the trajectory in the periodic extension of the wedges.

$\sqrt{m_1 v^2 + m_2 v^2} = 2\sqrt{m}v$ , where  $v$  is the initial velocity. In the final state, the rescaled speed is  $w'_2 = \sqrt{m}v'_2$ . Therefore  $v'_1 = 0$  and  $v'_2 = 2v$ .

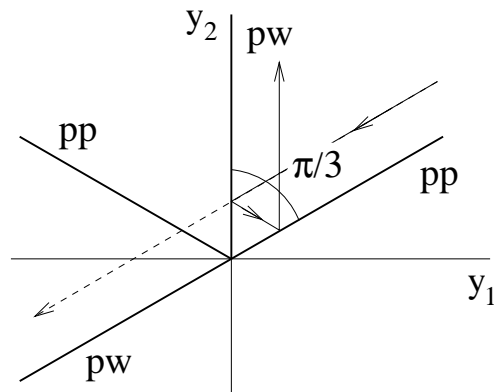


FIG. 5: Allowed wedge in  $y_i$  coordinates for the baseball-basketball system with  $m_1 = 3m_2$ . The dashed line again shows the billiard trajectory in the periodic extension of the wedges.

As a byproduct of the billiards approach, notice that as soon as  $m_1/m_2 < 3$ , the total angle of three wedges is less than  $180^\circ$  and there necessarily must be one more  $pw$  collision. Whenever the final trajectory is tangent to either a  $pp$  or a  $pw$  line, a critical point is defined where the total number of collisions changes by one. As  $m_1$  continues to decrease, a sequence of transitions arises. Each transition occurs when the wedge angle decreases below  $\pi/n$ , with  $n$  an integer. At this point the total number collisions increases from  $n-1$  to  $n$ . We therefore find that three collisions first occur when  $m_1 < 3m_2$ , four when  $m_1 < m_2$ , five when  $m_1 < 0.5278m_2$ , six when  $m_1 < m_2/3$ , *etc.*

#### V. THREE PARTICLES ON AN INFINITE LINE

The billiards approach gives an extremely simple way to solve a classic elastic collision problem that was appar-

ently first posed by Sinai [7]. Consider a three-particle system on an infinite line that consists of two approaching cannonballs, each of mass  $M$ . Between them (and non-symmetrically located) lies a ping-pong ball of mass  $m \ll M$ . Due to the collisions between the cannonballs and the intervening ping-pong ball, the latter rattles back and forth with a rapidly increasing speed until its momentum is sufficient to drive the cannonballs apart (Fig. 6). In the final state, the three particles are receding from each other. How many collisions occur before this final state is reached?

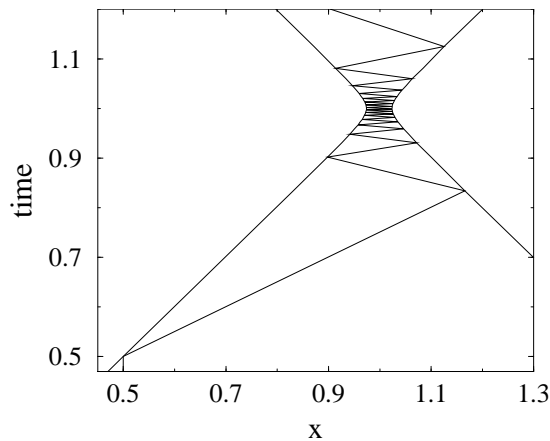


FIG. 6: Space-time diagram of the typical evolution of two cannonballs (heavy lines) approaching an initially stationary ping-pong ball (light line). The cannonballs each have mass  $M = 1$  and initial conditions  $(x_1(0), v_1(0)) = (0, 1)$  and  $(x_3(0), v_3(0)) = (2, -1)$ . A ping-pong ball of mass  $m = 0.005$  is initially at  $x_2(0) = 1/2$ . There are 31 collisions in total before the three particles recede. The first 30 collisions are shown.

Using energy and momentum conservation, we can determine the state of the system after each collision and thereby find the number of collisions before the three particles mutually recede. However, this approach is complicated and provides minimal physical insight (see Appendix and also [14, 15, 18, 19]). We now present a much simpler solution by mapping the original three-particle system onto a billiard in an appropriately-defined domain.

We denote the coordinates of the particles as  $x_1$ ,  $x_2$ , and  $x_3$ , with  $x_1 < x_2 < x_3$ . This order constraint between the particles again translates to a geometrical constraint on the accessible region for the billiard ball in the three-dimensional  $x_i$  space. Similarly, the trajectories of the particles on the line translate to the trajectory  $(x_1(t), x_2(t), x_3(t))$  of a billiard ball in the allowed region.

As in the previous examples, we introduce the rescaled coordinates  $y_i = x_i \sqrt{m_i}$ . These coordinates then satisfy the constraints

$$\frac{y_1}{\sqrt{M}} < \frac{y_2}{\sqrt{m}}, \quad \frac{y_2}{\sqrt{m}} < \frac{y_3}{\sqrt{M}}.$$

(The generalization to arbitrary masses is straightfor-

ward and is made in the next section.) In  $y_i$  space, the constraints correspond, respectively, to the effective billiard ball being confined to the half-space to the right of the plane  $y_1/\sqrt{M} = y_2/\sqrt{m}$  and to the half-space to the left of the plane  $y_2/\sqrt{m} = y_3/\sqrt{M}$ , as illustrated in Fig. 7. This defines the allowed region as an infinite wedge of opening angle  $\alpha$ .

The use of rescaled coordinates ensures that all collisions between the effective billiard particle and these constraint planes are specular. Further, momentum conservation gives

$$Mv_1 + mv_2 + Mv_3 = \sqrt{M}w_1 + \sqrt{m}w_2 + \sqrt{M}w_3 = 0,$$

where, without loss of generality, we take the total momentum to be zero. In this zero momentum reference frame, the trajectory of the billiard ball is always perpendicular to the diagonal,  $\vec{d} = (\sqrt{M}, \sqrt{m}, \sqrt{M})$ . Thus we may reduce the three-dimensional billiard to a two-dimensional system in the plane perpendicular to  $\vec{d}$ .

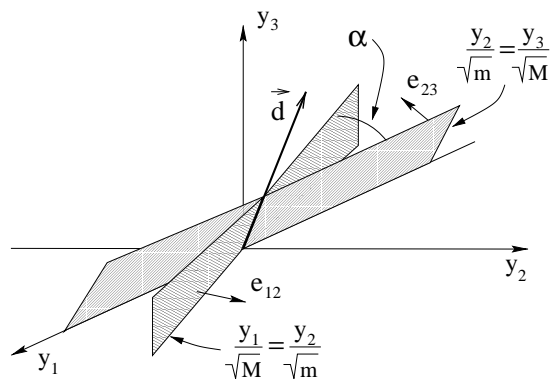


FIG. 7: Allowed wedge in the  $y_i$  coordinate system for a system of two cannonballs and an intervening ping-pong ball on an infinite line. The wedge is the region where the vectors  $e_{12}$  and  $e_{23}$  point towards.

To complete this picture, we need to find the wedge angle  $\alpha$ . The normals to the two constraint planes are  $\vec{e}_{12} = (-\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{m}}, 0)$  and  $\vec{e}_{23} = (0, -\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{M}})$ . Consequently, the angle between these planes is given by

$$\begin{aligned} \alpha &= \cos^{-1} \left( \frac{-\vec{e}_{12} \cdot \vec{e}_{23}}{|\vec{e}_{12}| |\vec{e}_{23}|} \right) \\ &= \cos^{-1} \left( \frac{1}{1 + \frac{m}{M}} \right). \end{aligned} \quad (5)$$

In the limit  $m/M \rightarrow 0$ , this gives  $\alpha \approx \sqrt{2m/M}$ . Finally, the maximum number of possible collisions is determined by the number of wedges that fit into the half plane. This gives

$$N_{\max} = \frac{\pi}{\alpha} \approx \pi \sqrt{\frac{M}{2m}}. \quad (6)$$

For  $m/M \rightarrow 0$  the opening angle of the wedge goes to zero and correspondingly, the number of collisions diverges.

## VI. THREE PARTICLES ON A RING AND THE TRIANGULAR BILLIARD

Finally, let us consider three elastically colliding particles of arbitrary masses  $m_1$ ,  $m_2$ , and  $m_3$  on a finite ring of length  $L$  [8, 9]. If we make an imaginary cut in the ring between particles 1 and 3, then we can write the order constraints of the three particles as

$$x_1 < x_2, \quad x_2 < x_3, \quad x_3 < x_1 + L$$

As usual, we employ the rescaled coordinates  $y_i = x_i \sqrt{m_i}$  to ensure that all collisions of the billiard ball with the domain boundaries in the  $y_i$  coordinates are specular. In these coordinates, the first two constraints again confine the particle to be between the planes defined by the normal vectors  $\vec{e}_{12} = (-\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, 0)$  and  $\vec{e}_{23} = (0, -\frac{1}{\sqrt{m_2}}, \frac{1}{\sqrt{m_3}})$ . Without the offset of  $L$ , the constraint  $x_3 < x_1 + L$  corresponds to a plane that slices the  $y_1$ - $y_3$  plane and passes through the origin. The offset of  $L$  means that we must translate this plane by a distance  $L\sqrt{m_3}$  along  $y_3$ . The fact that  $x_3$  is the lesser coordinate also means that the billiard ball is confined to the near side of this constraint plane. Thus the billiard ball must remain within a triangular bar whose outlines are shown in Fig. 8.

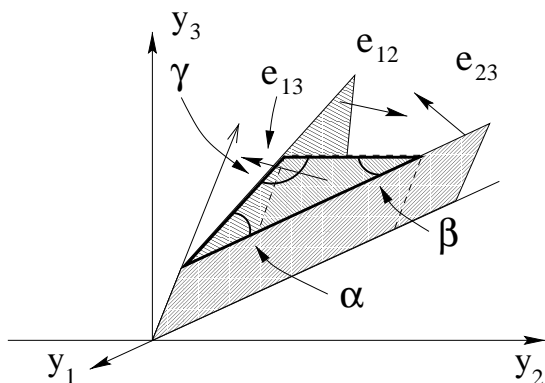


FIG. 8: Allowed region in the  $y_i$  coordinates for three particles of arbitrary masses on a ring of circumference  $L$ . The triangular billiard with angles  $\alpha$ ,  $\beta$  and  $\gamma$  is defined by the thick solid lines.

If the total momentum of the system is zero, then  $(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}) \cdot (w_1, w_2, w_3) = 0$  and the trajectory of the billiard ball remains within a triangle perpendicular to the long axis of the bar, with angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . We compute these angles by the same approach given in Eq. (5). Thus, for example,

$$\begin{aligned} \alpha &= \cos^{-1} \left( -\frac{\vec{e}_{12} \cdot \vec{e}_{23}}{|\vec{e}_{12}| |\vec{e}_{23}|} \right) \\ &= \cos^{-1} \left( \sqrt{\frac{m_2 m_3}{(m_1 + m_2)(m_2 + m_3)}} \right). \end{aligned} \quad (7)$$

The angles  $\beta$  and  $\gamma$  can be obtained by cyclic permutations of this formula.

Therefore the elastic collisions of three particles on a finite ring can be mapped onto the motion of a billiard ball within a triangular billiard table. One can then exploit the wealth of knowledge about triangular billiards [5, 6, 8] to infer basic collisional properties of the three-particle system. For example, periodic or ergodic behavior of the billiard translates to periodic or non-periodic behavior in the three-particle collision sequence.

## VII. DISCUSSION

We have shown how to recast the elastic collisions of point particles in one dimension into the motion of a billiard ball that moves at constant speed in a confined region of a higher-dimensional space. A crucial step in this reformulation is to introduce the rescaled coordinates  $y_i = x_i \sqrt{m_i}$ . This rescaling ensures that all collisions of the billiard ball with the boundaries of its accessible region are specular. For the examples of two particles on a semi-infinite line and a reflecting wall and for three particles on an infinite line, the allowed region for the billiard ball is an infinite wedge. For three particles on a ring, the allowed region is a triangular billiard. The shape of the associated wedge or triangle is readily calculable in terms of the particle masses in the original system.

When these masses are widely disparate, the opening angle of the wedge or one angle in the triangle becomes small. If a billiard ball enters such an acute corner, a large number of bounces occurs before the ball recedes from this corner. These frequent bounces are completely equivalent to a straight line trajectory passing close to the tips of a large number of periodic extensions of the wedge over a short distance. In the original system, either picture corresponds to a large number of collisions between neighboring particles.

The mapping onto a billiard system can, in principle, be generalized to an arbitrary number of particles  $N$ . The spatial dimension of the accessible region in the corresponding billiard is now  $N - 1$ -dimensional. While less is known about such high dimensional billiards, this mapping provides a useful perspective to deal with the elastic collisions of many particles in one dimension.

Finally, it is worth mentioning that a similar wedge mapping has been applied to determine the probability that three diffusing particles on the line obey various constraints on their relative positions [20]. In both the collisional and diffusive systems, the order constraint leads to nearly identical wedge constructions, and these provide elegant solutions to the original respective problems.

## VIII. ACKNOWLEDGMENTS

I thank T. Antal, S. Glashow, P. Hurtado, and P. Krapivsky for pleasant discussions, and P. Hurtado for helpful manuscript comments. I am grateful to NSF grant DMR0227670 for partial support of this work.

## APPENDIX A: PING-PONG BALL BETWEEN TWO CANNONBALLS: DIRECT SOLUTION

We consider three particles with masses  $m_i$ , positions  $x_i$  and velocities  $v_i$  for  $i = 1, 2, 3$ . Define the relative coordinates  $z_i = x_i - x_{i+1} < 0$  and the relative velocities  $v_{i,i+1} = v_i - v_{i+1}$ . From elementary mechanics, these relative velocities transform as follows (see Fig. 9):

$$\text{12 collision : } \begin{cases} v'_{12} = -v_{12} \\ v'_{23} = v_{23} + \lambda_{12}v_{12}, \end{cases} \quad (\text{A1})$$

$$\text{23 collision : } \begin{cases} v'_{12} = v_{12} + \lambda_{23}v_{23} \\ v'_{23} = -v_{23}, \end{cases} \quad (\text{A2})$$

with  $\lambda_{12} = 2m_1/(m_1 + m_2)$  and  $\lambda_{23} = 2m_3/(m_2 + m_3)$ . Once again, we can view the particle collisions as equivalent to the motion of a billiard ball in the third quadrant of the  $z$ -plane, but with non-specular reflections at each boundary.

It is convenient to characterize a trajectory by its polar angle  $\tan \theta = v_{23}/v_{12}$ . Then the above collision rules can be written as

$$\begin{aligned} \text{12 collision : } \tan \theta_n &= -\lambda_{12} - \tan \theta_{n-1}, \\ \text{23 collision : } \cot \theta_{n-1} &= -\lambda_{23}^{-1} - \cot \theta_{n-2}, \end{aligned} \quad (\text{A3})$$

where  $\theta_n$  is the angle after the  $n^{\text{th}}$  collision with the boundary. In writing these recursions, we use the fact that the 12 and 23 collisions alternate. Initially, the billiard ball is heading toward the corner, but eventually it “escapes” by having a trajectory with polar angle in the range  $(\pi, 3\pi/2)$ . This condition means that the three particles are all receding from each other. For a 12 collision, the incidence angle is in the range  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$  while the outgoing angle is in the range  $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ . In this case, escape means that  $\tan \theta > 0$ . Similarly, for a 23 collision, the incidence angle is in the range  $0 < \theta < \pi$  while the outgoing angle is in the range  $\pi < \theta < 2\pi$ . For this case, escape means that  $\cot \theta > 0$ .

To solve Eqs. (A3), we first write  $\theta_n$  in terms of  $\theta_{n-2}$ :

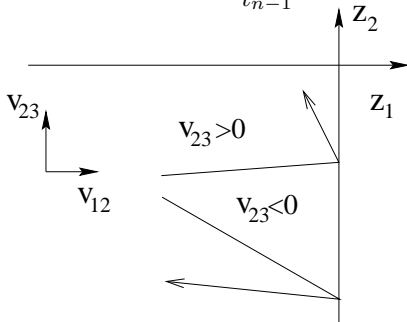
$$\tan \theta_n = -\lambda_{12} + \frac{1}{\frac{1}{\lambda_{23}} + \frac{1}{\tan \theta_{n-2}}}. \quad (\text{A4})$$

Next, let  $t_n = \tan \theta_n / \sqrt{\lambda_{12}\lambda_{23}}$ . This simplifies Eq. (A4) to

$$t_n = -\mu + \frac{1}{\mu + \frac{1}{t_{n-2}}}, \quad (\text{A5})$$

where  $\mu = \sqrt{\lambda_{12}/\lambda_{23}}$ . This equation can be written even more simply as

$$t_n = -\mu - \frac{1}{t_{n-1}}. \quad (\text{A6})$$



To solve this recursion formula, we define  $t_n \equiv g_n/h_n$  and find that Eq. (A6) is equivalent to the two first-order recursion relations  $h_n = g_{n-1}$  and  $g_n = -\mu g_{n-1} - h_{n-1}$ . This, in turn, is equivalent to the second-order recursion

$$g_n = -\mu g_{n-1} - g_{n-2}. \quad (\text{A7})$$

The general solution is  $g_n = A_+ \alpha_+^n + A_- \alpha_-^n$ , where  $A_{\pm}$  are constants and  $\alpha_{\pm} = (-\mu \pm \sqrt{\mu^2 - 4})/2$ . To complete the solution, we need an initial condition. For simplicity, we start with a billiard with incidence angle  $\theta = 0$  that has undergone a single 12 collision. This situation corresponds to the initial condition  $t_1 = -\mu$ . Imposing this condition, and performing some simple algebra, we find

$$t_n = \frac{\alpha_+^{n+1} - \alpha_-^{n+1}}{\alpha_+^n - \alpha_-^n}. \quad (\text{A8})$$

It is more convenient to write this in complex form by defining  $\alpha_{\pm} = A e^{\pm i\phi}$ . This then leads to

$$t_n = \frac{\sin(n+1)\phi}{\sin n\phi}, \quad (\text{A9})$$

where  $\phi = \tan^{-1} \sqrt{(4 - \mu^2)/\mu^2}$ .

The particles are all receding when  $t_n$  first becomes negative. The maximum number of collisions until this occurs is thus given by the condition  $t_n = 0$ ; this gives  $(n+1)\phi = \pi$ , or  $n \approx \pi/\phi$ . In the limit  $m_2/m_1 \equiv \epsilon_1 \rightarrow 0$  and  $m_2/m_3 \equiv \epsilon_3 \rightarrow 0$ , we have

$$\begin{aligned} \mu^2 &= \frac{4}{(1 + \epsilon_1)(1 + \epsilon_3)} \\ &= 4(1 - \epsilon_1 - \epsilon_3). \end{aligned}$$

Finally  $\phi = \tan^{-1} \sqrt{(4 - \mu^2)/\mu^2} \approx \sqrt{\epsilon_1 + \epsilon_3}$ . This then gives

$$N_{\max} \approx \frac{\pi}{\sqrt{\epsilon_1 + \epsilon_3}} = \pi \sqrt{\frac{m_1 m_3}{m_2(m_1 + m_3)}}. \quad (\text{A10})$$

In the special case of  $m_1 = m_3 = M$  and  $m_2 = m$ , this expression reduces to Eq. (6).

- 
- [1] D. Halliday, R. Resnick, and J. Walker, *Fundamental of Physics, Vol. 1*, 5th ed. (Wiley, New York, 1997), example 2.18.
- [2] J. Walker, *The Flying Circus of Physics* (J. S. Wiley & Sons, New York, 1977).
- [3] G. Galperin and A. Zemlyakov, *Mathematical Billiards* (in Russian) (Nauka, Moscow, 1990).
- [4] V. V. Kozlov and D. V. Treshshëv, *Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts* (Amer. Math. Soc., Providence, R.I., 1991).
- [5] S. Tabachnikov, *Billiards* (Société Mathématique de France; Amer. Math. Soc., Providence, R.I., 1995).
- [6] E. Gutkin, “Billiard in Polygons: Survey of Recent Results”, *J. Stat. Phys.* **81**, 7–26 (1996).
- [7] This problem was apparently first posed by Sinai. See *e.g.*, Ya. Sinai, *Introduction to Ergodic Theory*, (Princeton Univ. Press, Princeton, N.J., 1978).
- [8] S. L. Glashow and L. Mittag, “Three Rods on a Ring and the Triangular Billiard”, *J. Stat. Phys.* **87**, 937–941 (1996).
- [9] S. G. Cox and G. J. Ackland, “How Efficiently do Three Pointlike Particles Sample Phase Space?”, *Phys. Rev. Lett.* **84**, 2362–2365 (2000).
- [10] J. Rouet, F. Blasco, and M. R. Feix, “The One-Dimensional Boltzmann Gas: The Ergodic Hypothesis and the Phase Portrait of Small Systems”, *J. Stat. Phys.* **71**, 209–224 (1993).
- [11] See *e.g.*, A. Dhar, “Heat Conduction in a One-Dimensional Gas of Elastically Colliding Particles of Unequal Masses”, *Phys. Rev. Lett.* **86**, 3554–3557 (2001); P. L. Garrido, P. I. Hurtado, and B. Nadrowski, “Simple One-Dimensional Model of Heat Conduction Which Obeys Fourier’s Law” *Phys. Rev. Lett.* **86**, 5486–5489 (2001); O. Narayan and S. Ramaswamy, “Anomalous Heat Conduction in One-Dimensional Momentum-Conserving Systems”, *Phys. Rev. Lett.* **89**, 200601 (2002). For a recent review, see S. Lepri, R. Livi, and A. Politi, “Thermal Conduction in Classical Low-Dimensional Lattices”, *Phys. Repts.* **377**, 1–80 (2003).
- [12] For a general review, see “Granular Solids, Liquids, and Gases”, H. M. Jaeger, S. R. Nagel, and R. B. Behringer, *Rev. Mod. Phys.* **68**, 1259–1273 (1996).
- [13] S. McNamara and W. R. Young, “Inelastic Collapse and Clumping in a One-Dimensional Granular Medium”, *Phys. Fluids A* **4**, 496–504 (1992); I. Goldhirsch and G. Zanetti, “Clustering Instability in Dissipative Gases”, *Phys. Rev. Lett.* **70**, 1619–1622 (1993); S. McNamara and W. R. Young, “Dynamics of a Freely Evolving, Two-Dimensional Granular Medium”, *Phys. Rev. E* **53**, 5089–5100 (1996).
- [14] A similar approach to that in the Appendix is given in P. Constantin, E. Grossman, and M. Mungan, “Inelastic Collision of Three Particles on the Line as a Two-Dimensional Billiard”, *Physica D* **83** 409–420 (1995). See also, T. Zhou and L. P. Kadanoff, “Inelastic Collapse of Three Particles”, *Phys. Rev. E* **54**, 623–628 (1996).
- [15] M. Hasegawa, “Broken Ergodic Motion of Two Hard Particles in a One-Dimensional Box”, *Phys. Lett. A* **242**, 19–24 (1998); B. Cipra, P. Dini, S. Kennedy, and A. Kolan, “Stability of One-Dimensional Inelastic Collision Sequences of Four Balls”, *Physica D* **125**, 183–200 (1999).
- [16] B. Bernu and R. Mazighi, “One-Dimensional Bounce of Inelastically Colliding Marbles on a Wall”, *J. Phys. A* **23**, 5745–5754 (1990).
- [17] N. D. Whelan, D. A. Goodings, and J. K. Cannizzo, “Two Balls in One Dimension with Gravity”, *Phys. Rev. A* **42**, 742–754 (1990).
- [18] D. W. Jepsen, “Dynamics of a Simple Many-Body System of Hard Rods”, *J. Math. Phys.* **6**, 405–413 (1965).
- [19] T. J. Murphy, “Dynamics of Hard Rods in One Dimension”, *J. Stat. Phys.* **74**, 889–901 (1994).
- [20] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).