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Anomalous voltage distribution of random resistor networks and a new model for the backbone at the percolation threshold

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We develop a new approach for studying the random resistor network by focusing on the distribution of voltage drops across each bond We introduce a simple model which provides a useful description of the percolating backbone, and which shows that the voltage distribution is log normal, with an infinite set of exponents required to describe the voltage moments. This latter prediction is verified by simulations of a resistor network on the square lattice.

The critical behavior of the random resistor network is a classical percolation problem which has received considerable theoretical attention.[1] In this Rapid Communication, we suggest that new insights into the problem can be gained by considering the distribution of voltage drops across each bond in the network. Both model calculations and numerical simulations on a two-dimensional network suggest that the voltage distribution cannot be described by a scaling function of the conventional form. Consequently, the exponents characterizing the moments of this distribution are not simply related; that is, constant "gap" scaling fails. This is reminiscent of the recent findings of Harris and coworkers[2] for the moments of the resistive susceptibility.

The distribution of voltage drops in an electrical network is of central importance as the second moment equals the network conductivity. Moreover, we shall show that this distribution gives detailed microscopic information, analogous to the cluster-size distribution in percolation, which aids in quantifying the structure of the percolating cluster. Finally, the voltage distribution is analogous to the distribution of strains in a random elastic network. Thus, knowledge of the distribution may ultimately provide insight into the mechanical failure of a variety of random media, such as gels, unwoven textiles, or fractured rocks.

In order to discuss the voltage distribution analytically, we first introduce an extremely simple hierarchical lattice[3] (HL) model which successfully describes the properties of the backbone of the critical percolating cluster (PC) for any spatial dimension d. We then show that for the HL, the voltage distribution is a log binomial. This leads to novel scaling properties for the voltage moments which are then verified by numerical simulations.

The model is illustrated in Fig. 1 for $d=2$. Starting with a single bond of unit resistance, this bond is replaced by the series and parallel combination, or "unit cell," of unit resistance bonds shown to the right. This construction is iterated indefinitely, with the Nth level being achieved by replacing each bond in the $(N-1)$th level by the unit cell.

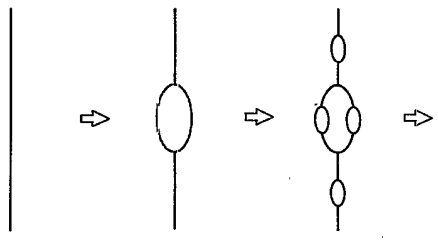

FIG. 1. Hierarchical lattice for the case $\lambda=1$, corresponding to $d=2$.

The resulting structure is a self-similar fractal made of singly connected bonds, termed links, and multiply connected blobs.[4,5] A link is defined as one which, if cut, would render the network disconnected, while the remaining bonds comprise the blobs. For this model, we can easily calculate critical quantities such as the number of bonds N_B, the average number of steps in the set of all self-avoiding walks (SAW's), $N_{\rm SAW}$, the resistance R, and the number of links L_1 between the extreme ends of the HL. For example, one can easily check that for an N-level hierarchy, $N_B=4^N$, $N_{\rm SAW}=3^N$, $R=2.5^N$, and $L_1=2^N$. We can now eliminate the unphysical quantity N in favor of L_1: $N_B=L_1^{\zeta_B}$, $N_{\rm SAW}=L_1^{\zeta_{\rm SAW}}$, $R=L_1^{\zeta_R}$, with $\zeta_B=2$, $\zeta_{\rm SAW}=1.58\ldots$, and $\zeta_R=1.322\ldots$. To compare with the exponents of PC's, we note that in percolation one usually expresses the critical quantities in terms of the Euclidean distance L. Namely, $N_B\sim L^{d_f^{\rm BB}}$, $N_{\rm SAW}\sim L^{\tilde{\zeta}_{\rm SAW}}$, $R\sim L^{\tilde{\zeta}_R}$, and $L_1\sim L^{1/\nu}$,[5] where $d_f^{\rm BB}$ is the fractal dimension of the backbone, $\tilde{\zeta}_R$ is the resistance exponent which is related to the conductivity exponent $\tilde{t}$ by $\tilde{t}=d-2+\tilde{\zeta}_R$, and ν has been proved[5] to coincide with the connectedness-length exponent. Eliminating L in favor of L_1, we obtain for PC's in $d=2$, $\zeta_B=d_f^{\rm BB}=2.11\ldots$, $\zeta_{\rm SAW}=\nu\tilde{\zeta}_{\rm SAW}=1.73\ldots$, $\zeta_R=\nu\tilde{\zeta}_R=1.297\ldots$,[1] in good agreement with the HL exponents.

The HL can be generalized to model PC's in d dimensions by introducing the parameter $\lambda=2/L_1$, which is the ratio of the number of bonds in a blob to the number of links within a unit cell, by choosing $\lambda=(6-d)/4$ for $2\leq d\leq 6$, $\lambda=d-1$ for $1\leq d-\leq 2$, and $\lambda=0$ for $d>6$. This choice is rather arbitrary; however, it is a simple linear expression in d which has the property of being zero for $d=1$ and 6, and attaining a maximum for $d=2$. This reproduces an essential feature of the links and blobs structure of PC's; blobs are irrelevant in $d=1$ and 6,[5,6] and are relatively most important in $d=2$. With this simple choice for λ, the resulting exponents are close to the best numerical exponent estimates for PC's (Table I), thus lending confidence in the validity of the HL model (Ref. 9).

We now turn to the distribution of voltage drops. If a unit voltage is applied across an Nth-order hierarchical lattice, the number of bonds biased by a voltage

TABLE I. Comparison of the exponents from the hierarchical model with the best available estimates for the percolation exponents discussed in the text.

	ζ_{BB}		ζ_R		ζ_{SAW}	
d	Data	Model	Data	Model	Data	Model
2	2.11[a]	2.0	1.297[b]	1.322	1.73[c]	1.58
3	1.61[a]	1.57	1.14[d]	1.175	1.32[c]	1.32
4	1.25[a]	1.29	1.07[d]	1.085	1.11[c]	1.16
5	1.11[a]	1.11	1.02[d]	1.029	1.05[c]	1.06
6	1.0	1.0	1.0	1.0	1.0	1.0

[a]Reference 5. [c]Reference 7.
[b]Reference 1. [d]Reference 8.

$V(k),n(V(k))$, is equal to

$$n(V(k)) = (2/\lambda)^N \lambda^k \binom{N}{K} , \quad (1)$$

where the voltage $V(k)$ takes on the discrete set of values $[\lambda/(\lambda+4)]^N 2^k$, with $k=0,1,2,\ldots,N$, and for general λ, $L_1=(2/\lambda)^N$. The maximal voltage drop $V_{\max} = V(k=N)$ occurs across the links where all the current I in the network flows. Applying Ohm's law to a link of unit conductivity gives $V_{\max}=I$. Since a unit voltage is applied across the network, it follows that the network conductance G equals the total current flowing. Very generally, then, for a network containing links, such as a random resistor network at the percolation threshold, we have the striking result, $G=V_{\max}$. Thus, we may write

$$V_{\max} \sim L_1^{-\zeta_R} . \quad (2)$$

Thus, from the voltage distribution we obtain (i) the number of links from $n(V_{\max})=L_1$, (ii) the number of backbone bonds from $\sum_V n(V) \sim L_1^{\zeta_B}$, and (iii) the resistance exponent ζ_R from the second moment of $n(V)$ which equals the conductance G, i.e., $\sum_V V^2 n(V) \sim L_1^{-\zeta_R}$.

We can easily calculate higher moments of the voltage distribution from Eq. (1), and we find

$$\langle V^k \rangle = \sum_V V^k n(V) \sim L_1^{-p_k} , \quad (3)$$

where

$$p_k = k - 1 + [k \ln(1+\lambda/4) - \ln(1+\lambda/2^k)]/(\ln 2/\lambda) ,$$

and we can identify $-p_0$ with ζ_B and p_2 with ζ_R. Thus a very interesting feature predicted by our model is that an infinite set of exponents is required to describe the moments, $\langle V^k \rangle$. Moreover, the average value $V_{av} = \langle V \rangle / \sum_V n(V) \sim L_1^{-p_{av}}$ and the most probable value of the voltage $V_{mp} \sim \exp[(\ln V)_{av}] \sim L_1^{-p_{mp}}$ scale differently. For example, for the hierarchical lattice with $\lambda=1 (d=2)$, $p_{av}=1.736\ldots$, while $p_{mp}=3.015\ldots$. However, we do expect constant-gap scaling in the averages $[(\ln V)^k]_{av}$ which we find to vary as $(\ln L_1)^{-k}$ with $\Delta=1$.

The origin of a nonconstant gap exponent, i.e., $p_k - p_{k-1} \neq$ const, and the different scaling laws for V_{mp} and V_{av} becomes more apparent if we write the voltage distribution after approximating the binomial factor by a Gaussian. For example, for $\lambda=1$, since $k=\ln V/\ln 2$ and $N=\ln L_1/\nu \ln 2$, we obtain

$$n(V) \cong L_1^{\zeta_B} \left(\frac{2\ln 2}{\ln L_1}\right)^{1/2} \exp\left[\frac{(\ln V - \ln V_{mp})^2}{[(\ln 2 \ln L_1)/2]}\right] , \quad (4)$$

with $n(V)=0$ for $V > V_{\max}$. From Eq. (4), the nonconstant gap in the $\langle V^k \rangle$ stems from $n(V)$ being a scaling function of $\ln V$ and $\ln L_1$ rather than of V and L_1 itself. This continuum limit is not a very useful approximation, however, because of the predominant role that the cutoff in $n(V)$ plays in the calculation of the moments $\langle V^k \rangle$.

The higher moments of the voltage distribution can also be related to an infinite family of lengths with which one can quantify the structure of the percolating backbone. We know that the backbone is made of links and blobs with the total number of backbone bonds diverging as $L_1^{\zeta_B}$. To characterize the remaining bonds of the backbone, we first define $\mathcal{N}(\alpha)$ as the number of bonds in which a fraction $\alpha = V/V_{\max}$ of the total network current flows. This is related to the voltage distribution through $\mathcal{N}(\alpha) = n(\alpha V_{\max})$. Next define the family of lengths $\mathcal{L}_k$ and associated exponents ζ_k through

$$\mathcal{L}_k = \sum \alpha^k \mathcal{N}(\alpha) \sim L_1^{\zeta_k} , \quad (5)$$

which can be related to the voltage distribution by $\mathcal{L}_k = V_{\max}^{-k} \sum_V n(V) V^k$. From Eqs. (2), (3), and (5) we obtain for the HL,

$$\zeta_k = k p_2 - p_k = 1 + \ln(1+\lambda/2^k)/[\ln(2/\lambda)] .$$

Then $\mathcal{L}_0$ coincides with the total number of bonds in the backbone, $\mathcal{L}_\infty$ coincides with the number of links (i.e., the number of bonds with $\alpha=1$), and $\mathcal{L}_2$ scales as the resistance. In general, $\mathcal{L}_k$ is a length measure of the backbone in which each bond is weighted by a factor α^k. Clearly $\mathcal{L}_k \geq \mathcal{L}_{k+1}$ and $\zeta_k \geq \zeta_{k+1}$. In one and six dimensions the ζ_k all coincide with 1, since the blobs are irrelevant, and constant gap scaling holds. On the other hand, the maximum dispersion in the ζ_k occurs in $d=2$, where the blobs are relatively most important, leading to the maximum departure from constant gap scaling.

In order to test the HL model predictions for the voltage

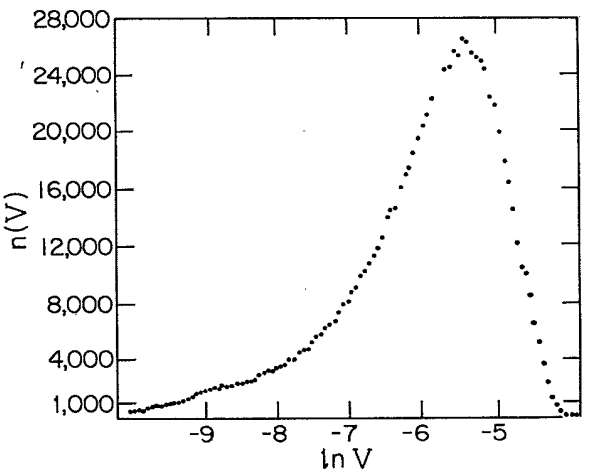

FIG. 2. The voltage distribution for 1100 realizations of a 100×100 lattice at p_c plotted vs $\ln V$.

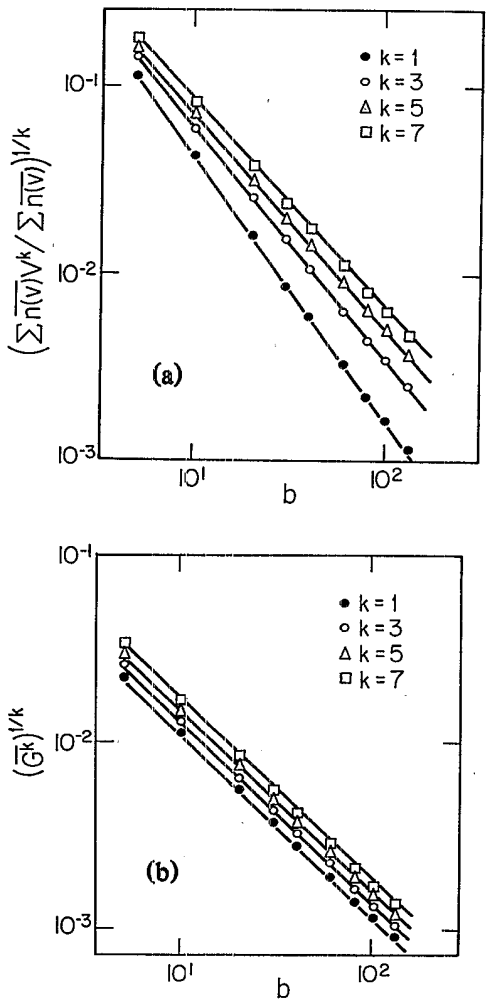

FIG. 3. (a) Double logarithmic plot of $[\sum_V \overline{n(V)}V^k/\sum_V \overline{n(V)}]^{1/k}$, where the overbar refers to configurational averaging, showing non-constant-gap scaling, and (b) $(\overline{G}^k)^{1/k}$, showing constant-gap scaling.

distribution, we have performed numerical simulations for a random resistor network at the percolation threshold (bond concentration 50%) on a $L \times L$ square lattice for the range $L=5$ (50 000 configurations) to $L=130$ (1000 configurations). A unit potential drop is imposed across the sample, and free boundary conditions in the transverse direction are used. As indicated in Fig. 2, the distribution averaged over all configurations is more usefully visualized in terms of $\ln V$ rather than in terms of V, where 100 bins on a logarithmic scale are displayed. In Fig. 3, several moments of this distribution are plotted versus L. The differing slopes of the straight line fits to the data confirm that each moment is governed by an independent exponent as predicted by the HL model. More strikingly, the estimated values of the exponents from the simulation agree closely with those predicted from the HL (Table II), indicating the general utility of the model for describing even microscopic aspects of the random resistor network. We have also computed numerically the configurational averages of the conductance and its inverse, $\overline{G}^k$ and $\overline{G}^{-k^{-1}}$. Both quantities appear to scale as $L^{k\zeta_R/\nu}$, with ζ_R/ν estimated to be 0.98 ± 0.02, in agreement with the most recent numerical results for ζ_R,[1] suggesting that constant gap scaling does occur for the distribution of conductances.

TABLE II. Estimates for the exponents p_k and $\zeta_k = kp_2 - p_k$.

k	p_k Data	p_k Model	ζ_k Data	ζ_k Model
0	-2.18 ± 0.02	-2.00	2.16	2.00
1	-0.28 ± 0.02	-0.263	1.58	1.585
2	1.31 ± 0.02	1.3219	1.30	1.322
3	2.78 ± 0.03	2.7959	1.12	1.17
4	4.19 ± 0.04	4.2003	1.01	1.09

In conclusion, we have introduced the distribution of voltage drops as a basic quantity to characterize the conductivity problem and the structure of the percolating backbone. A simple hierarchical model has been introduced from which (a) accurate values for exponents of percolation in $d=1-6$ can be found easily, and (b) an infinite set of exponents is required to describe the scaling properties of the moments of the voltage distribution, and this set can be interpreted in terms of the structure of the backbone. In the limiting case where only links remain, i.e., $d=1$ or 6, the infinite set reduces to a single gap exponent equal to $1/\nu$. This new type of critical phenomena is observed in our simulations of the two-dimensional random resistor network.

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