*Work*, (University of Chicago, Chicago, 1978). For a historical discussion that disagrees in part with Drake we suggest the paper by R. H. Naylor, "Galileo and the Problem of Free Fall," Brit. J. Hist. Sci. **7**, 105–134 (1974).

[3]Actually the error is slightly "worse" than the factor 5/7, because Galileo used a groove cut in his inclined plane so as to keep the ball centered laterally. Instead of a single contact point of ball and plane at sphere radius $R$ below the center, the ball thus has two contact points, at the sides of the groove, giving an effective "rolling radius" $R'$, with $R'<R$. The formulas then involve both $R$ and $R'$. We omit these corrections, for simplicity, and because in the experiments we describe there was no need for a groove. For the correction, see Ref. 10.

[4]Reference 2, pp. 178–179.

[5]Reference 2, p. 185.

[6]Stillman Drake, "The Role of Music in Galileo's Experiments," Sci. Am. 98–104 (June, 1975). To reproduce a "metronome rate" Galileo sang a favorite song at its "correct" tempo. Drake replicated Galileo's experiments. To compare short time intervals as the ball rolls down the incline Drake used moveable rubber band "frets" that gave an audible "click" each time the ball rolls over a fret. Drake argues convincingly that Galileo used similar moveable frets, adjusting their position until the clicks were evenly spaced in time, thus discovering that (in our notation) $d\propto t^2(H/L)$.

[7]The weights are tied on the string at $y=n^2d_1$, with $n=0,1,2,3,4$, etc., above the floor at $y=0$, giving sounds at equal intervals $t_1=(2d_1/g)^{1/2}$. See R. M. Sutton, *Demonstration Experiments in Physics* (McGraw-Hill, New York, 1938), who suggests $d_1=2$ in., which gives $t_1=0.102$ s, which we find unappealing—slightly too fast to vocalize, and with no readily available Japanese standard chirper for comparison. G. D. Freier and F. J. Anderson, *A Demonstration Handbook of Physics* (American Association of Physics Teachers, 1972, 1981) suggest $d_1=1$ ft, giving $t_1=1/4$ s. Excellent! Easily vocalized, and easily compared with standard Japanese wrist watch alarms, and easily memorized! R. D. Edge, *String & Sticky Tape Experiments* (American Association of Physics Teachers, College Park, MD, 1987) suggests $d_1=6$ in., giving $t_1=0.177$ s. Why bother? Not worth memorizing. Immersed as we all are in a sea of Japanese digital beepers teaching us and forever reminding us of the sound of eight chirps per s, I much prefer $d_1=3$ in. and $t_1=1/8$ s, so that by comparison with their digital wrist watch alarms the students can accurately estimate the magnitude $g=32$ ft per s per s. (This is one case where feet and inches are better than cgs or MKS units.) Students should use their digital alarms to memorize the sound of eight chirps per s (or four chirps per s for some alarms) just as they should memorize the length of their own natural pace, for estimating distances of a few 100 m. A nice demo in a stair well uses seven weights tied to the string at distances $y$ above the floor given, with $d=3$ in.$=1/4$ ft, by (note the "spaces" with "missing" weights!)

$y=d,-,9d,16d,25d,-,49d,-,-,-,121d,-,169d.$

The resulting sound pattern is recognized by everyone to be the rhythm of a familiar ditty. We let the reader supply the well known words. A nice project would be to use as weights for this ditty seven objects that (when they hit the floor) emit the right pitches.

[8]Stillman Drake, "Galileo's Experimental Confirmation of Horizontal Inertia," ISIS. **64**, 290–305 (1973); Stillman Drake and James MacLachlan, "Galileo's Discovery of the Parabolic Trajectory," Sci. Am. 102–110 (March, 1975). These two papers are practically the same, except for an important statement in the Sci. Am. paper, "When we first analyzed Galileo's data in 1972, one of us (Drake) [in the ISIS paper] believed the inclined plane that Galileo used for the experiment recorded in f.116 was probably tilted at an angle of 64 degrees to the table, ... Now, however, we believe that, for that experiment ... Galileo employed a plane at an angle of only 30 degrees to the table." Drake and MacLachlan don't explain why they changed Drake's mind, but we agree that the slope must have been gentle, because of the quantitative agreement with "rolling without slipping," which could occur for slope 30 deg but not 64 deg.

[9]Stillman Drake, "Galileo's Discovery of the Law of Free Fall," Sci. Am. 85–92 (May, 1973).

[10]D. E. Shaw and F. J. Wunderlich, "Study of the slipping of a rolling sphere," Am. J. Phys. **52** (11), 997–1000 (1984). These authors use a channel to guide the ball laterally and hence include corrections involving $R'/R$. See Ref. 3.

[11]Pietro Redondi, *Galileo, Heretic* (Princeton U.P., Princeton, NJ, 1987).

# Life and death in an expanding cage and at the edge of a receding cliff

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The survival probabilities of a particle diffusing within an expanding "cage" and near the edge of a receding "cliff," with death occurring when the diffuser reaches a boundary of the system, are investigated. Especially interesting behavior arises when the position of the boundary recedes from the diffuser as $\sqrt{At}$. In this case, the recession matches the rms displacement $\sqrt{Dt}$ with which diffusion tends to bring the diffuser to its demise. For both the cage and cliff problems, the survival probability $S(t)$ exhibits a nonuniversal power-law decay in time, $S(t)\sim t^{-\beta}$, in which the value of $\beta$ is dependent on the detailed properties of the boundary motion. Heuristic approaches are applied for the cases of "slow" ($A/D\ll 1$) and "fast" ($A/D\gg 1$) boundary motion which yield approximate expressions for $\beta$. An asymptotic analysis of the survival probability for the cage and cliff problems is also performed. The approximate expressions for $\beta$ are in good agreement with the exact results for nearly the entire range of possible boundary motions. 

## I. INTRODUCTION

Consider a particle which diffuses within a one-dimensional "cage" $[-L(t),\ L(t)]$ and is absorbed, or dies, whenever it touches the walls (Fig. 1). We are interested in determining the probability for such a "prisoner" to survive until time $t, S(t)$. The behavior of this survival probability is a simple example of a first-passage problem whose general

  

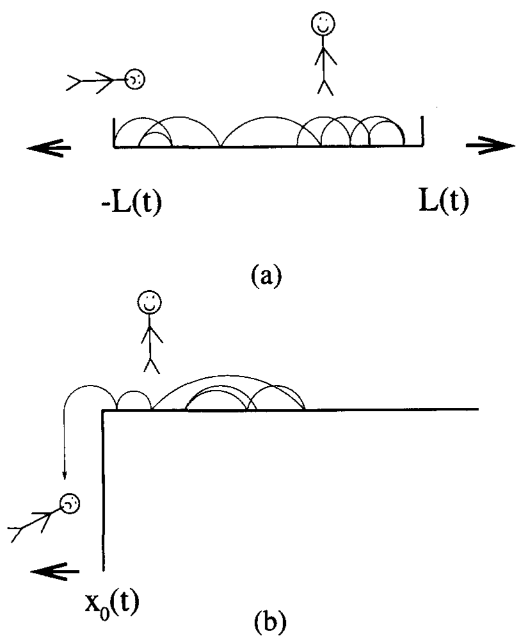


Fig. 1. (a) The prisoner in the expanding cage and (b) the daredevil at the edge of a receding cliff.

understanding is a basic aspect of stochastic processes.[1–5] The methodology of solving first-passage problems and the resulting behavior both provide fundamental insights and have applications to a wide variety of stochastic problems in physics, chemistry, and biology.

In a cage of fixed length $2L$, it is well known that the survival probability decays as $\exp(-\pi^2 Dt/4L^2)$ in the long-time limit.[2–5] More interesting behavior arises when we aid the prisoner by allowing the cage walls to recede by choosing $L(t)=(At)^\alpha$. (Notice that the units of $A$ depend on $\alpha$ in order that the right-hand side of this relation has units of length.) The possibility of cage growth obviously increases the particle lifetime and can also dramatically change the form of the survival probability. The problem has been extensively investigated in both the physics[6,7] and mathematics literature.[8–11] However, the methodology used in these papers is generally more complex and less intuitive than that presented here. It is possible to obtain a fairly comprehensive understanding of the asymptotic behavior of the system through simple-minded approximations which are based on the limiting cases of a slowly expanding[12,13] or rapidly expanding cage. Our goal in this paper is to present these intuitively motivated treatments of the problem. These approaches, given in Sec. II, are suitable for advanced undergraduates or beginning graduate students interested in learning about the simplest consequences of a moving boundary on a classical boundary value problem.[14–16]

In an expanding cage, there are three regimes of behavior that are determined by the competition between the rate at which the cage grows and the rate at which diffusion brings the particle to the cage walls:

(i) For $\alpha<\frac{1}{2}$, the cage grows more slowly than the rms displacement of a freely diffusing particle. This slowly moving boundary value problem can be solved with the adiabatic approximation that is often first discussed in beginning graduate courses on quantum mechanics.[12,13] Roughly speaking, the probability distribution of the particle in the long-time limit can be accurately approximated by the form appropriate for a fixed-size cage, but with a time-dependent boundary condition. This leads to a survival probability which decays as a stretched exponential in time, $S(t)\propto\exp(-t^\gamma)$, with $0<\gamma<1$ and $\gamma\to 1$ in the limit that the cage walls are immobile ($\alpha\to 0$).

(ii) For $\alpha>\frac{1}{2}$, the cage grows more rapidly than the rms displacement of a freely diffusing particle and the probability distribution of the particle can be well approximated by that for free diffusion. With this form for the probability distribution, one finds that the survival probability goes to a nonzero limiting value as $t\to\infty$.

For both cases (i) and (ii), the existence of established methods to determine the time dependence of $S(t)$ is based on the fact that the solution is, in some sense, "close" to that of either a particle in a fixed cage [for case (i)], or a freely diffusing particle [for case (ii)].

(iii) For the marginal situation of $\alpha=\frac{1}{2}$, richer and more interesting behavior arises in which the competition between the cage length $L(t)=(At)^{1/2}$ and the diffusion length $(Dt)^{1/2}$ plays a fundamental role. When $A<D$, it turns out that it is still possible to apply the adiabatic approximation, i.e., the probability density is "close" to that in the case where the cage walls are static. Conversely, for a rapidly expanding cage, $A>D$, the "free" particle approximation turns out to be appropriate. These two approximations predict that the particle survival probability decays as a power law in time, $S(t)\sim t^{-\beta}$, but with the exponent $\beta$ dependent on the dimensionless ratio $A/D$. In the context of critical phenomena, such behavior is known as nonuniversality. This is in contrast to the conventional universal behavior, where the characteristic exponents of power-law divergences do not depend on microscopic details of the system.

Boundary value problems with the boundary displacement growing as $\sqrt{t}$ arise naturally in the "Stefan problem." This is a classical description for the motion of the interface between two phases, e.g., liquid and solid, when the system is undergoing a transition between these two phases.[14–16] For example, when the air of an air–water interface is maintained at a fixed temperature below 0 °C while the water away from the interface remains at 0 °C, a growing layer of ice forms whose thickness grows as $\sqrt{t}$. The nature of heat diffusion near the moving boundary is an essential ingredient in determining the interface motion. The approaches of this paper are useful for solving this type of problem. It is also worth mentioning that moving boundary value problems are notoriously difficult to solve in general.[14–16] However, when boundaries move at the same rate as diffusion, namely as $\sqrt{t}$, a scaling approach, as presented in Sec. III, often permits an exact solution to the problem.

To ascertain the accuracy of our heuristic approaches, we also present an asymptotic analysis of the diffusion equation with the moving boundary condition corresponding to a growing cage in Sec. III. This approach shows that the asymptotic probability density may be written in terms of parabolic cylinder functions. From this description, we verify that the survival probability has a nonuniversal power law decay with time, $S(t)\sim t^{-\beta}$, with the exponent $\beta$ dependent on the ratio $A/D$. We also find that the exponents given by the above two heuristic treatments approach the corresponding exact values in the limits of $A/D\to 0$ and $A/D\to\infty$. This degree of accuracy is satisfying, but somewhat surprising, as the adiabatic and free particle approximations are uncontrolled. That is, there does not exist a quantitative procedure for assessing the errors in these approximations, nor a systematic method for improving accuracy. However, simple-minded asymptotic methods of the spirit of the adiabatic and free particle approximations often work much better than one might anticipate.

We also consider the related problem of a "daredevil" which diffuses in the one-dimensional semi-infinite domain $x>x_0(t)$, and falls to its death whenever the "cliff" at $x=x_0(t)$ is reached. When the cliff location is fixed, it is well known that the survival probability of the particle is $S(t) \equiv \int_{x_0}^{\infty} c(x,t)dt \propto t^{-1/2}$ (Ref. 4). Thus the particle is sure to fall off the cliff (although its mean lifetime is infinite). This same decay law continues to hold if the cliff recedes slowly, i.e., $x_0 = -(At)^{\alpha}$ with $\alpha<\frac{1}{2}$.[8–11] Conversely, if the cliff recedes rapidly compared to the particle diffusion, i.e., $\alpha>\frac{1}{2}$, then there is a finite probability for the particle to survive. In the extreme case where the cliff recedes at constant velocity $v$, then as $t\to\infty$, the survival probability has the limiting value $S(t\to\infty) = 1 - e^{-v\ell_0/D}$. Here $\ell_0$ is the initial distance from the daredevil to the cliff. When the cliff recedes at the same rate as the particle diffuses, $x_0(t) = -\sqrt{At}$ with $A$ of the order of $D$, marginal behavior again arises in which the survival probability exhibits a nonuniversal power-law decay in time. These results can be obtained by a straightforward adaptation of the methods developed for the cage problem and nicely illustrates their general utility.

## II. HEURISTIC APPROACHES FOR PRISONER SURVIVAL IN AN EXPANDING CAGE

Formally, the time evolution of the survival probability is governed by the diffusion equation for the underlying concentration, $c(x,t)$, at position $x$ at time $t$,

$$\frac{\partial c(x,t)}{\partial t} = D\,\frac{\partial^2 c(x,t)}{\partial x^2}, \tag{1}$$

where $D$ is the (constant) diffusion coefficient. For the cage geometry, this equation is defined on the domain $-L(t)\leq x\leq L(t)$, subject to the absorbing boundary condition $c(x=\pm L(t),t)=0$. The absorbing boundary condition imposes the death of the prisoner whenever he or she touches the cage walls. For simplicity, we generally consider the initial condition $c(x,t=0)=\delta(x)$. With this type of an initial condition whose spatial integral is normalized to unity, the concentration $c(x,t)$ coincides with the probability distribution function. Correspondingly, the survival probability $S(t)$ is given by the spatial integral of the probability density,

$$S(t)\equiv \int_{-L(t)}^{L(t)} c(x,t)dx. \tag{2}$$

Because of the absorbing boundary condition, only those diffusing trajectories which do not touch the cage walls are included in this integral.

For a fixed-length cage $[-L, L]$, the solution to the diffusion equation (1) may be written as an eigenfunction expansion in which each eigenmode decays exponentially in time, and with a different characteristic decay time. In the long time limit, only the most slowly decaying eigenmode remains and the density approaches the asymptotic form

$$c(x,t)\to \text{const.} \times e^{-D\pi^2 t/4L^2} \cos\left(\frac{\pi x}{2L}\right). \tag{3}$$

The proportionality constant in this expression depends on the initial conditions, and the higher-order terms in the eigenfunction expansion have been dropped. From this form for $c(x,t)$, the survival probability decays exponentially in time.

Now suppose that the cage expands slowly, that is, the time dependence of $L(t)$ is such that $L(t) \ll \sqrt{Dt}$. In this case, the adiabatic approximation indicates that the density distribution approaches the same form as in the fixed-cage case, except that the parameters in this probability distribution acquire time dependence to satisfy the moving boundary condition.[12,13] Within this approximation, the corresponding probability density is

$$c(x,t)\approx c_{\text{adiab}}(x,t)=f(t)\cos\left(\frac{\pi x}{2L(t)}\right), \tag{4}$$

with the amplitude $f(t)$ to be determined. Substituting Eq. (4) into Eq. (1) leads to

$$\dot{f} = -\left(\frac{D\pi^2}{4L^2}\right)f - \left(\frac{\pi x}{2L^2}\right)\tan\left(\frac{\pi x}{2L}\right)\dot{L}f. \tag{5}$$

When $L(t)$ grows as $(At)^{\alpha}$ with $\alpha<\frac{1}{2}$, the second term on the right-hand side may be neglected and the controlling factor,[17] i.e., the most rapidly varying factor as a function of time in the full expression for the amplitude $f(t)$, is given by

$$\exp\left(-\frac{\pi^2 D}{4}\int_0^t dt'\, L^{-2}(t')\right). \tag{6}$$

The full asymptotic behavior of $f(t)$ is expected to contain a power-law prefactor in $L$. However, this prefactor is not easily accessible with our naive approach, although it naturally emerges from the rigorous treatment following Eq. (15), below.

Making use of Eq. (4), we compute the survival probability to be

$$S(t)\approx \int_{-L(t)}^{L(t)} c_{\text{adiab}}(x,t)dx = \frac{4}{\pi} f(t)L(t). \tag{7}$$

Thus for $\alpha<\frac{1}{2}$ the controlling factor of the survival probability decays as a stretched exponential in time:

$$\exp\left[-\frac{\pi^2 D}{4(1-2\alpha)A^{2\alpha}}\, t^{1-2\alpha}\right]. \tag{8}$$

The full asymptotic behavior contains an additional power-law factor $\sqrt{L} \sim t^{\alpha/2}$ [see the discussion following Eq. (15)]. Notice that the above controlling factor could have been guessed (but without power law prefactors and also with an incorrect numerical factor inside the exponential) merely by substituting $L(t)\propto t^{\alpha}$ into the survival probability of a particle in a fixed size cage, Eq. (3).

For the marginal case of $\alpha=\frac{1}{2}$, the second term in Eq. (5) is ostensibly no longer negligible. However, when $A<D$, the cage still grows more slowly than the rms displacement of free diffusion. Thus the approximation made by neglecting the second term in Eq. (5) might be expected to be reasonable, especially in the limit $A\ll D$. With this condition, then the following nonuniversal power-law behavior for the survival probability is found:

$$S(t)\approx t^{-\beta}, \quad \text{with } \beta\approx \frac{\pi^2 D}{4A}. \tag{9a}$$

In writing this exponent value, an additive factor of $\frac{1}{2}$, which stems from the contribution of $L(t)$ in Eq. (7), has been ignored because it is negligible as $A/D\to 0$. Another, more substantive, reason for ignoring this additive factor is that corrections to $\beta$ of the same order arise when the contribution of the second term in the governing equation for $f(t)$ in Eq. (5) are included. When this is done, one finds the following more accurate expression for $\beta$, in which all terms of order unity have been included:

$$\beta=\frac{\pi^2 D}{4A}-\frac{1}{4}. \tag{9b}$$

The difference between the above two predictions of $\beta$ is relatively small when $A/D<1$ and justifies *a posteriori* the extension of the adiabatic approximation to the situation of a marginally growing cage $L(t)=\sqrt{At}$, as long as $A<D$.

In the complementary case of a rapidly growing cage, where the time dependence of $L(t)$ is such that $L(t)\gg\sqrt{Dt}$ (either $\alpha>\frac{1}{2}$ or $\alpha=\frac{1}{2}$ and $A\gg D$), it is plausible to assume that the particle density distribution approaches a Gaussian but with a survival probability, i.e., the spatial integral of $c(x,t)$ within $[-L(t), L(t)]$, which decays in time. Thus we hypothesize that

$$c(x,t)\approx c_{\text{free}}(x,t)=\frac{S(t)}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{10}$$

Although this distribution does not satisfy the absorbing boundary condition, the inconsistency should be negligible, since the density is exponentially small at the cage walls. The decay of the mass may now be found by equating the flux of probability to the cage walls, $-2D\partial c/\partial x$, to the mass loss within the cage. By computing the spatial integral of $c(x,t)$ in the cage, it immediately follows that the survival probability approaches a constant for $\alpha>\frac{1}{2}$. However, in the marginal case of $\alpha=\frac{1}{2}$, equating the mass loss to the flux gives

$$\dot{S}=-\frac{S}{t}\sqrt{\frac{A}{4\pi D}}\exp\left(-\frac{A}{4D}\right). \tag{11}$$

This again leads to power law behavior for the survival probability, $S\sim t^{-\beta}$, with

$$\beta=\sqrt{\frac{A}{4\pi D}}\exp\left(-\frac{A}{4D}\right). \tag{12}$$

The predictions of this "free particle" approximation become progressively more accurate as $A/D\to\infty$. In this limit, the particle diffuses more slowly than the cage expands and there is a finite probability to survive asymptotically, as reflected by the fact that $\lim_{A/D\to\infty}\beta=0$.

## III. ASYMPTOTIC ANALYSIS FOR THE MARGINALLY GROWING CAGE

Let us now investigate more carefully the borderline case where $L(t)=\sqrt{At}$. Within a scaling formulation, it is natural to hypothesize that the density can be written in terms of the dimensionless variables

$$\xi\equiv\frac{x}{L(t)} \quad\text{and}\quad \sigma\equiv\frac{x}{\sqrt{Dt}}.$$

Since both the cage length and the diffusion length grow at the same rate, these scaling variables coincide up to a constant numerical factor $\rho=\xi/\sigma=\sqrt{A/D}$. It proves convenient to consider $\xi$ as the basic variable, with $-1\leq\xi\leq 1$, and treat $\rho$ as a dimensionless parameter which also characterizes the solution. Thus we seek solutions for the concentration of the form,

$$c(x,t)\sim t^{-\beta-1/2}\mathscr{C}_\rho(\xi), \tag{13}$$

with $\mathscr{C}_\rho(\xi)$ a scaling function. The power-law prefactor is chosen to ensure that the survival probability decays as $t^{-\beta}$, as defined previously.

Substituting Eq. (13) into Eq. (1), we find that the scaling function $\mathscr{C}_\rho(\xi)$ satisfies the ordinary differential equation

$$\frac{D}{A}\frac{d^2\mathscr{C}}{d\xi^2}+\frac{1}{2}\xi\frac{d\mathscr{C}}{d\xi}+\left(\beta+\frac{1}{2}\right)\mathscr{C}=0. \tag{14}$$

For notational simplicity, we have dropped the redundant $\rho=\sqrt{A/D}$ dependence, since dependence of the scaling function on $A/D$ already appears through the explicit appearance of $\beta$ in Eq. (14). Introducing now $\xi=\eta\sqrt{2D/A}$ and $\mathscr{C}(\xi)=e^{-\eta^2/4}\mathscr{D}(\eta)$ transforms Eq. (14) into the canonical form for the parabolic cylinder equation:[18]

$$\frac{d^2\mathscr{D}}{d\eta^2}+\left[2\beta+\frac{1}{2}-\frac{\eta^2}{4}\right]\mathscr{D}=0. \tag{15}$$

A spatially symmetric solution to Eq. (15) (appropriate for a particle starting in the middle of the cage) is

$$\mathscr{D}(\eta)=\tfrac{1}{2}(\mathscr{D}_{2\beta}(\eta)+\mathscr{D}_{2\beta}(-\eta)), \tag{16}$$

with $\mathscr{D}_\nu(\eta)$ the parabolic cylinder function of order $\nu$. Finally, the relation between the decay exponent $\beta$ and $A/D$ is determined by the absorbing boundary condition

$$\mathscr{D}_{2\beta}\left(\sqrt{\frac{A}{2D}}\right)+\mathscr{D}_{2\beta}\left(-\sqrt{\frac{A}{2D}}\right)=0. \tag{17}$$

One can easily determine the limiting behaviors of $\beta=\beta(A/D)$ for $A/D\to 0$ and $A/D\to\infty$ and thus check the validity of the heuristic predictions given in Eqs. (9) and (12), respectively. In the former case, the exponent $\beta$ is large and, hence, the second two terms in the parentheses in Eq. (15) can be neglected. Therefore, the distribution approaches the cosine form in this limit and Eq. (9) provides the correct asymptotics. In the latter case, $A\gg D$, $\beta\to 0$ and Eq. (15) approaches the Schrödinger equation for the ground state of the harmonic oscillator, for which $\mathscr{D}_0(\eta)=\exp(-\eta^2/4)$. For small but finite $\beta$ it is natural to seek a perturbative solution:

$$\mathscr{D}(\eta)=\exp(-\eta^2/4)+\beta\mathscr{A}(\eta)+\cdots . \tag{18}$$

Substituting this expansion into Eq. (15) yields an inhomogeneous linear equation for the correction $\mathscr{A}(\eta)$:

$$\frac{d^2\mathscr{A}}{d\eta^2}+\left(\frac{1}{2}-\frac{\eta^2}{4}\right)\mathscr{A}=-2e^{-\eta^2/4}. \tag{19}$$

Introducing $\mathscr{B}(\eta)$ through $\mathscr{A}(\eta)=\exp(-\eta^2/4)\mathscr{B}(\eta)$, we obtain $\mathscr{B}''-\eta\mathscr{B}'=-2$. By solving this latter equation, the perturbative solution for $\mathscr{D}(\eta)$ is

$$\mathscr{D}(\eta)=e^{-\eta^2/4}-2\beta e^{-\eta^2/4}\int_0^\eta d\eta_1\, e^{\eta_1^2/2}\int_0^{\eta_1}d\eta_2\, e^{-\eta_2^2/2}+O(\beta^2). \tag{20}$$

Combining Eq. (20) and the absorbing boundary condition $\mathscr{D}(\sqrt{A/2D})=0$ one can reproduce (after a straightforward but lengthy computation) the asymptotics given by Eq. (12).

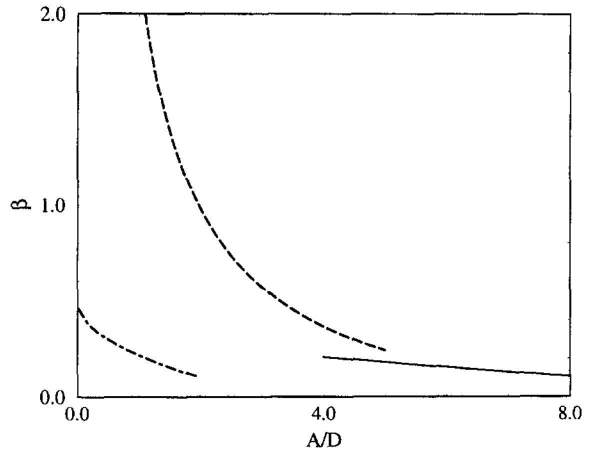


Fig. 2. Plot of the limiting values of the exponent $\beta$ as a function of $A/D$ for the cage (dashed) and cliff (dot-dashed) problems based on the adiabatic approximation, which is valid only for $A/D \ll 1$, and the exponent $\beta$ for both the cage and cliff problems based on the free approximation, which is valid only for $A/D \gg 1$ (solid). These plots pertain to the marginal case where $L(t)=(At)^{1/2}$.

This provides rigorous justification for the previous heuristic predictions for the decay exponent $\beta$.

It is also interesting to consider the mean lifetime of the particle, $\langle t\rangle=\int_0^\infty dt\, S(t)$. [This result can be obtained from the straightforward definition of the mean lifetime, $\langle t\rangle=\int_0^\infty t'\mathscr{F}(t')dt'$, where $\mathscr{F}(t)$ is the probability for the particle to be absorbed at time $t$. Integrating this relation by parts and using $S(t)=1-\int_0^t\mathscr{F}(t')dt'$, the above formula for $\langle t\rangle$ follows.] This mean lifetime is finite for $\beta>1$ and infinite for $\beta<1$. In the borderline case of $\beta=1$, Eq. (15) coincides with equation describing the second excited state of the simple harmonic oscillator. Thus it has the solution $\mathscr{D}(\eta) = (1-\eta^2)e^{-\eta^2/4}$.[12,13] Furthermore, the absorbing boundary condition now gives $A=2D$. Thus the borderline case between a finite and an infinite mean survival time corresponds to $A=2D$. It is gratifying that for this case of $A=2D$, the naive adiabatic approach gives $\beta=\pi^2/8\cong 1.234$, while the "improved" adiabatic approach [where the power-law prefactors in the asymptotic form of $c(x,t)$ are retained] gives $\beta=\frac{1}{8}\pi^2-\frac{1}{4}\cong 0.984$. The relatively small deviation from the value $\beta=1$ in both cases provides a sense for the accuracy of the adiabatic approach in the regime $A<2D$.

## IV. SURVIVAL NEAR THE EDGE OF A CLIFF

Let us now turn to the case of a diffusing particle which survives if it remains within the semi-infinite domain $x_0(t)\leq x<\infty$. As mentioned in the Introduction, the most interesting case is that of $x_0(t) = -\sqrt{At}$ as $t\to\infty$, corresponding to the cliff receding at the same rate at which diffusion tends to transport the particle to the cliff. This situation can again be analyzed by methods similar to those applied for the marginally growing cage problem.

It is convenient to first change variables from $(x,t)$ to $(x'=x-x_0(t),t)$ to fix the absorbing boundary at the origin. Thus the initial diffusion equation is transformed to the convection–diffusion equation (where the prime is now dropped)

$$\frac{\partial c}{\partial t}-\frac{x_0}{2t}\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}, \quad \text{for } 0\leq x<\infty, \tag{21}$$

with the *fixed* absorbing boundary condition $c(x=0,t)=0$. Notice also that this equation also describes *biased diffusion* with a leftward directed drift whose "velocity" equals $x_0/2t$. As in the case of the marginally expanding cage, we introduce the dimensionless length $\xi=-x/x_0$ and apply the same scaling assumption, Eq. (13), for the probability density in the cliff geometry. Substituting this form into Eq. (21) gives

$$\frac{D}{A}\frac{d^2\mathscr{C}}{d\xi^2}+\frac{1}{2}(\xi-1)\frac{d\mathscr{C}}{d\xi}+\left(\beta+\frac{1}{2}\right)\mathscr{C}=0. \tag{22}$$

It is also expedient to transform from the quantities $\xi$ and $\mathscr{C}(\xi)$ to $\eta$ and $\mathscr{D}(\eta)$ (but in a slightly different way than that employed for the cage problem):

$$\xi-1=\sqrt{\frac{2D}{A}}\,\eta, \quad \mathscr{C}(\xi)=\exp\left(-\frac{\eta^2}{4}\right)\mathscr{D}(\eta). \tag{23}$$

With these definitions, we find that $\mathscr{D}(\eta)$ satisfies the same parabolic cylinder equation [Eq. (15)] as in the cage problem. However, slightly different boundary conditions apply. Death of the particle at the edge of the cliff implies

$$\mathscr{D}(-\sqrt{A/2D})=0. \tag{24a}$$

On the other hand, $S(t)\equiv\int_0^\infty dx\,c(x,t)\leq 1$ implies the boundary condition at $\eta=\infty$,

$$\mathscr{D}(\eta=\infty)=0. \tag{24b}$$

Mathematically, the determination of $\beta$ and $\mathscr{D}(\eta)$ is equivalent to finding the ground state energy and wave function of a quantum particle in a potential composed of an infinite barrier at $\eta=-\sqrt{A/2D}$ and the harmonic oscillator potential for $\eta>-\sqrt{A/2D}$. Higher excited states do not contribute in the long time limit. A general solution to Eq. (22) satisfying $\mathscr{D}(\infty)=0$ is

$$\mathscr{D}(\eta)=\mathscr{D}_{2\beta}(\eta), \tag{25}$$

and the absorbing boundary condition $\mathscr{D}_{2\beta}(-\sqrt{A/2D})=0$ determines the relation between the decay exponent $\beta$ and $A/D$.

Since Eq. (25) provides only an implicit relation $\beta=\beta(A/D)$, it is useful to determine the limiting behaviors of $\beta$ for small and large values of $A/D$. These relations can be derived directly from Eq. (22) and elementary facts about the quantum mechanics of the harmonic oscillator, rather than relying on mathematical properties of the parabolic cylinder functions. In the limit of a slowly moving boundary, $A\ll D$, the wall is close to the origin, when expressed in terms of the variable $\eta$. When the wall is exactly at the origin, the ground state of the truncated potential is obviously the first excited state for pure harmonic oscillator, namely, $2\beta=1$ and $\mathscr{D}(\eta)=\eta\exp(-\eta^2/4)$. For $A\ll D$, we therefore expect that $0<1-2\beta\ll 1$. This again suggests the perturbative solution,

$$\mathscr{D}(\eta)=\eta\exp(-\eta^2/4)+(1-2\beta)\mathscr{A}(\eta)+\cdots. \tag{26}$$

Substituting this expansion into the parabolic cylinder equation yields for $\mathscr{A}(\eta)$

$$\frac{d^2\mathscr{A}}{d\eta^2}+\left(\frac{3}{2}-\frac{\eta^2}{4}\right)\mathscr{A}=\eta e^{-\eta^2/4}. \tag{27}$$

Introducing $\mathscr{B}(\eta)$ through $\mathscr{A}(\eta)=\eta\exp(-\eta^2/4)\mathscr{B}(\eta)$, we find $\mathscr{B}''-(\eta-2/\eta)\mathscr{B}'=1$. Solving this equation subject to the boundary condition Eq. (24b) gives $\mathscr{B}(\eta)$, from which one ultimately obtains

$$\mathscr{D}(\eta)=\eta e^{-\eta^2/4}+(1-2\beta)\eta e^{-\eta^2/4}\times\int_\eta^\infty d\eta_1\ \eta_1^{-2}e^{\eta_1^2/2}\int_{\eta_1}^\infty d\eta_2\eta_2^2\ e^{-\eta_2^2/2}+\cdots. \tag{28}$$

Finally, imposition of the absorbing boundary condition Eq. (24a) gives

$$\beta=\frac{1}{2}-\sqrt{\frac{A}{4\pi D}}. \tag{29}$$

One can treat the opposite limit $A\gg D$ similarly. In terms of the coordinate $\eta$, the location of the wall goes to $-\infty$. Hence the unperturbed ground state for this system is just the ground state for the pure harmonic oscillator, namely, $\beta=0$ and $\mathscr{D}(\eta)=\exp(-\eta^2/4)$. Following the same perturbative approach as in the complementary case of $A\ll D$, we find

$$\mathscr{D}(\eta)=e^{-\eta^2/4}+2\beta e^{-\eta^2/4}\int_\eta^\infty d\eta_1\ e^{\eta_1^2/2}\int_{\eta_1}^\infty d\eta_2\ e^{-\eta_2^2/2}+O(\beta^2). \tag{30}$$

Combining Eq. (30) with the absorbing boundary condition one finds the same expression for $\beta$, Eq. (12), as was found for a finite cage.

## V. SUMMARY AND DISCUSSION

We have presented both a heuristic and an asymptotic approach to determine the survival probability and the density distribution for (a) a diffusing "prisoner" in a growing cage, $[-L(t), L(t)]$, and (b) a diffusing "daredevil" in the domain $x>x_0(t)$ with a cliff at $x=x_0(t)$. We were primarily concerned with the "marginal" case where $L(t)=\sqrt{At}$ and $x_0=-\sqrt{At}$ (with $A$ of the order of $D$), so that boundary of the system recedes at the same rate at which diffusion tends to bring the diffuser toward its demise. In these marginal situations, the survival probability $S(t)$ of the diffuser exhibits a nonuniversal power-law decay in time, $S(t)\sim t^{-\beta}$, with $\beta$ dependent on $A/D$ (Fig. 2). The value of $\beta$ in the limiting case of $A\gg D$ for both the cage and cliff problems can be obtained by assuming free diffusion for the particle within its allowed domain, and then determining the survival probability in terms of the flux leaving the boundary of the system. In the complementary case of $A\ll D$, the survival probability can be obtained by an adiabatic approximation in which the probability distribution of the particle retains the same form as that in the system with static boundaries. These limiting behaviors are found to coincide with the results from an asymptotic analysis of the underlying exact equation of motion.

For the cage problem, our results can be straightforwardly extended to general spatial dimension. Following the same adiabatic and "free particle" approximations that were applied in one dimension, we find that the decay exponent $\beta$ becomes

$$\beta=\begin{cases}\dfrac{j_d^2 D}{A} & \text{adiabatic;}\\[2ex] \dfrac{1}{\Gamma\left(\frac{d}{2}\right)}\left(\dfrac{A}{4D}\right)^{d/2}\exp\left(-\dfrac{A}{4D}\right) & \text{free.}\end{cases} \tag{31}$$

Here $j_d$ is the first positive root of the spherical Bessel function $x^{1-d/2}J_{d/2-1}(x)$, with $j_2\cong 2.40483$ and $j_3=\pi$, for example. Similarly, in the case of marginal cage growth, the $d$-dimensional analog of the scaling ansatz, Eq. (13), can be applied, leading to a straightforward generalization of Eq. (14). In three dimensions, in particular, the additional transformation $\mathscr{C}(\xi)=\tilde{\mathscr{C}}(\xi)/\xi$ leads to the same parabolic cylinder equation of Eq. (14) for $\tilde{\mathscr{C}}$, but with the parameter $\beta+\frac{1}{2}$ replaced by $\beta+1$.

It is instructive to compare the behavior of the survival probabilities given here with those of the related problem where the absorbing boundaries themselves undergo diffusive motion. These are situations for which exact solutions have been given previously.[19] First, consider the relatively trivial example of a diffusing particle near a cliff when the position of the cliff also diffuses with a diffusivity $A$. This situation is isomorphic to the case of a static cliff and a particle with diffusivity $D+A$. Thus the survival probability decays universally as $t^{-1/2}$. As might be expected, a cliff which is systematically receding from a diffusing particle with $L(t)\propto\sqrt{t}$ leads to a larger survival probability compared to the case of a stochastically moving cliff.

On the other hand, the survival of a diffusing particle inside a cage where both walls diffuse (each with diffusivity $A$) is more interesting. A variety of exact solutions show that the prisoner survival probability decays nonuniversally, $S(t)\sim t^{-\beta}$, with

$$\beta=\pi/2\ cos^{-1}(D/(D+A)).^{19}$$

For rapidly diffusing walls, $\beta\rightarrow 1$, while the corresponding limit for systematically receding walls is $\beta\rightarrow 0$. Clearly if the cage walls are receding rapidly, the particle is more likely to survive compared to the case where the cage walls are diffusing rapidly. In the latter case, there is the possibility of rapid motion of the cage walls *toward* the particle. On the other hand, in the limit of slowly diffusing walls, $\beta\rightarrow\sqrt{\pi^2D/8A}$. Strangely, this is almost the square root of the corresponding expression for $\beta$ quoted in Eq. (9). Intriguingly, the particle is more likely to survive in a stochastically and slowly growing cage than in a cage which grows systematically at the same average rate.

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[1] See, e.g., *Selected papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).

[2] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North–Holland, Amsterdam, 1981).

[3] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer-Verlag, Berlin, 1990).

[4] G. H. Weiss and R. J. Rubin, "Random walks: Theory and selected ap-

plications," Adv. Chem. Phys. **52**, 363–505 (1983), and references therein.
[5]D. T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic, Boston, 1992).
[6]L. Turban, "Anisotropic critical phenomena in parabolic geometries: The directed self-avoiding walk," J. Phys. A **25**, L127–L134 (1992).
[7]F. Iglói, "Directed polymer inside a parabola: Exact solution," Phys. Rev. A **45**, 7024–7029 (1992); F. Iglói, I. Peschel, and L. Turban, "Inhomogeneous systems with unusual critical behaviour," Adv. Phys. **42**, 683–740 (1993).
[8]L. Breiman, "First exit time from the square root boundary," Proc. Fifth Berkeley Symp. Math. Statist. Prob. **2**, 9–16 (1966).
[9]H. E. Daniels, "The minimum of a stationary Markov superimposed on a U-shape trend," J. Appl. Prob. **6**, 399–408 (1969).
[10]K. Uchiyama, "Brownian first exit from sojourn over one sided moving boundary and application," Z. Wahrsch. verw. Gebiete **54**, 75–116 (1980).
[11]P. Salminen, "On the hitting time and the exit time for a Brownian motion to/from a moving boundary," Adv. Appl. Prob. **20**, 411–426 (1988).
[12]See, e.g., A. Messiah, *Quantum Mechanics* (Interscience, New York, 1961).
[13]L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968).
[14]J. Crank, *Free and Moving Boundary Problems* (Clarendon, Oxford, 1984).
[15]H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U.P., Oxford, 1959), Chap. XI.
[16]R. Ghez, *A Primer of Diffusion Problems* (Wiley, New York, 1988), Chap. IV.
[17]For example, if a function has the asymptotic form $f(x)=x^\alpha \exp(x^\beta)(1+h_1(x)+h_2(x)+\cdots)$ as $x\to\infty$, with $h_i(x)\to 0$ as $x\to\infty$, then the *leading behavior* is $x^\alpha \exp(x^\beta)$ and the *controlling factor* is $\exp(x^\beta)$. See, e.g., p. 80 of Ref. 18.
[18]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw–Hill, New York, 1978); M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[19]F. Leyvraz, (unpublished 1987); D. ben-Avraham, "Computer simulation methods for diffusion-controlled reactions," J. Chem. Phys. **88**, 941–947 (1988); M. E. Fisher and M. P. Gelfand, "The reunions of three dissimilar vicious walkers," J. Stat. Phys. **53**, 175–189 (1988).

# Thomas precession and the Liénard–Wiechert field

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The role of the Thomas precession in the dynamic formation of the electric field lines of a moving charged particle is demonstrated. A simple derivation of the Thomas precession formula is given, based only on the Lorentz contraction of moving bodies. A simple and physically appealing construction is developed for determining points on a field line. Field line diagrams are generated and discussed. These diagrams vividly reveal the existence and magnitude of the Thomas precession. © *1996 American Association of Physics Teachers.*

## I. INTRODUCTION

The Thomas precession is one of those subtle counterintuitive consequences of special relativity whose existence students often find initially quite difficult to accept. Standard discussions[1–5] often involve fairly advanced concepts and techniques which tend to encourage the belief that the Thomas precession is an esoteric phenomenon whose origin and significance is hard to grasp. It will be shown here that the Thomas precession can be understood very simply in terms of the well-known Lorentz contraction of a moving body. Furthermore, its influence is manifested in and can easily be seen in plots of the electric field lines of a moving charged particle. The first of these insights is not new, but is included for completeness. An alternative simple derivation of the Thomas precession can be found in the appendix of Muller's paper.[5]

This paper extends the work of Purcell,[6] Tsien,[7] and others.[8,9] It provides a simple description of the mechanism by which the field lines are formed and explicitly reveals the role of the Thomas precession in their formation. A straightforward, intuitive derivation of the Thomas precession formula is given in Sec. II. This is followed, in Sec. III, by discussion and further development of a simple picture of the dynamical formation of the electric field lines of a moving charged particle. Section IV is devoted to the display and discussion of field lines for a charged particle undergoing uniform motion in a circle. These results directly illustrate the effect of the Thomas precession in a manner that needs no mathematics to appreciate. The insight achieved in this analysis is summarized in Sec. V.

## II. THOMAS PRECESSION FORMULA

Imagine three inertial reference frames, $S_0$, $S_1$, and $S_2$, such that $S_1$ and $S_2$ are moving at velocities $\boldsymbol{\beta}$ and $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$, respectively, relative to $S_0$, where $\boldsymbol{\beta}=\mathbf{v}/c$. We will be specifically interested in the limit $\delta\beta\ll 1$, so that frames $S_1$ and $S_2$ are moving nonrelativistically with respect to one another. Imagine a line of posts fixed in $S_0$ running parallel to $\boldsymbol{\beta}$ and another line running parallel to $\boldsymbol{\beta}+\delta\boldsymbol{\beta}$. With no loss of generality, we may choose the $x$ axis in $S_0$ parallel to $\boldsymbol{\beta}$ and the $y$ axis such that $\delta\boldsymbol{\beta}$ lies in the $xy$ plane.

Figure 1 shows the situation as seen in each of the three reference frames. In frame $S_0$, the two lines of posts are at rest and separated by a small angle $\delta\theta$. In frames $S_1$ and $S_2$

  