

4.3 (a) What fraction of the H_2 gas at sea level and at a temperature of 300 K can escape from the earth's gravitational field?

(b) Why do we still have H_2 gas in the atmosphere at sea level?

4.4 Using relativistic dynamics for gas molecules find, for a dilute gas of zero total momentum,

(a) the equilibrium distribution function;

(b) the equation of state.

Answer. PV is independent of the volume. Hence it is NkT by definition of T .

4.5 (a) Estimate the probability that a stamp (mass = 0.1 g) resting on a desk top at room temperature (300 K) will spontaneously fly up to a height of 10^{-8} cm above the desk top.

Hint. Think not of one stamp but of an infinite number of noninteracting stamps placed side by side. Formulate an argument showing that these stamps obey the Maxwell-Boltzmann distribution.

Answer. Let m = mass of stamps, h = height, g = acceleration of gravity. Probability $\approx e^{-mgh/kT}$

4.6 A room of volume $3 \times 3 \times 3 \text{ m}^3$ is under standard conditions (atmospheric pressure and 300 K).

(a) Estimate the probability that at any instant of time a 1-cm³ volume anywhere within this room becomes totally devoid of air because of spontaneous statistical fluctuations.

(b) estimate the same for a 1-Å³ volume.

Answer. Let N = total number of air molecules, V = volume of room, v = the volume devoid of air. Probability $\approx e^{-N(v/V)}$

4.7 Suppose the situation referred to in Problem 4.6a has occurred. Describe qualitatively the behavior of the distribution function thereafter. Estimate the time it takes for such a situation to occur again, under the assumption that molecular collisions are such that the time sequence of the state of the system is a random sequence of states.

4.8 (a) Explain why in (4.42) we arrived at the formula for the Maxwell-Boltzmann distribution for a gas with no average momentum ($\mathbf{p}_0 = 0$), although average momentum was not specified as a macroscopic condition in (4.35) and (4.36).

(b) Derive the Maxwell-Boltzmann distribution for a gas with average velocity \mathbf{v}_0 , using the method of the most probable distribution.

4.9 Let

$$H = \int d^3p f(\mathbf{p}, t) \log f(\mathbf{p}, t)$$

where $f(\mathbf{v}, t)$ is arbitrary except for the conditions

$$\int d^3p f(\mathbf{p}, t) = n$$

$$\frac{1}{n} \int d^3p \frac{p^2}{2m} f(\mathbf{p}, t) = \epsilon$$

Show that H is minimum when f is the Maxwell-Boltzmann distribution.

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C 5

TRANSPORT PHENOMENA

5.1 THE MEAN FREE PATH

To begin our discussion on the approach to equilibrium of a gas initially not in equilibrium, we introduce the qualitative concept of the mean free path and related quantities.

A gas is not in equilibrium when the distribution function is different from the Maxwell-Boltzmann distribution. The most common case of a nonequilibrium situation is that in which the temperature, density, and average velocity are not constant throughout the gas. To approach equilibrium, these nonuniformities have to be ironed out through the transport of energy, mass, and momentum from one part of the gas to another. The mechanism of transport is molecular collision, and the average distance over which molecular properties can be transported in one collision is the *mean free path*. It is the average distance traveled by a molecule between successive collisions. We give an estimate of its order of magnitude.

The number of collisions happening per second per unit volume at the point \mathbf{r} in a gas is given by

$$Z = \int d^3p_1 d^3p_2 d^3p'_1 d^3p'_2 \delta^4(P_f - P_i) |T_{fi}|^2 f(\mathbf{r}, \mathbf{p}_1, t) f(\mathbf{r}, \mathbf{p}_2, t)$$

where $f(\mathbf{r}, \mathbf{p}, t)$ is the distribution function. The integration over \mathbf{p}'_1 and \mathbf{p}'_2 can be immediately effected to yield

$$Z = \int d^3p_1 \int d^3p_2 \sigma_{\text{tot}} |\mathbf{v}_1 - \mathbf{v}_2| f(\mathbf{r}, \mathbf{p}_1, t) f(\mathbf{r}, \mathbf{p}_2, t) \quad (5.1)$$

A free path is defined as the distance traveled by a molecule between two successive collisions. Since it takes two molecules to make a collision, every collision terminates two free paths. The total number of free paths occurring per second per unit volume is therefore $2Z$. Since there are n molecules per unit volume, the average number of free paths traveled by a molecule per second is

$2Z/n$. The *mean free path*, which is the average length of a free path, is given by

$$\lambda = \frac{n}{2Z\bar{v}} \quad (5.2)$$

where $\bar{v} = \sqrt{2kT/m}$ is the most probable speed of a molecule. The average duration of a free path is called the *collision time* and is given by

$$\tau = \frac{\lambda}{\bar{v}} \quad (5.3)$$

For a gas in equilibrium, $f(\mathbf{r}, \mathbf{p}, t)$ is the Maxwell-Boltzmann distribution. Assume for an order-of-magnitude estimate that σ_{tot} is insensitive to the energy of the colliding molecules and may be replaced by a constant of the order of πa^2 where a is the molecular diameter. Then we have

$$\begin{aligned} Z &= \frac{\sigma_{\text{tot}}}{m} \int d^3p_1 \int d^3p_2 |\mathbf{p}_1 - \mathbf{p}_2| f(\mathbf{p}_1) f(\mathbf{p}_2) \\ &= \frac{\sigma_{\text{tot}} n^2}{m(2\pi mkT)^3} \int d^3p_1 \int d^3p_2 |\mathbf{p}_1 - \mathbf{p}_2| \exp\left[-\frac{p_1^2 + p_2^2}{2mkT}\right] \\ &= \frac{\sigma_{\text{tot}} n^2}{m(2\pi mkT)^3} \int d^3P \int d^3p |\mathbf{p}| \exp\left[-\frac{1}{kT}\left(\frac{P^2}{4m} + \frac{p^2}{m}\right)\right] \end{aligned}$$

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $\mathbf{p} = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$. The integrations are elementary and give

$$Z = 2n^2\sigma_{\text{tot}} \sqrt{\frac{kT}{\pi m}} = \sqrt{\frac{2}{\pi}} n^2\sigma_{\text{tot}} \bar{v} \quad (5.4)$$

Therefore

$$\begin{aligned} \lambda &= \frac{\sqrt{\pi/8}}{n\sigma_{\text{tot}}} \\ \tau &= \frac{\sqrt{\pi/8}}{n\sigma_{\text{tot}}\bar{v}} \end{aligned} \quad (5.5)$$

We see that the mean free path is independent of the temperature and is inversely proportional to the density times the total cross section.

The following are some numerical estimates. For H_2 gas at its critical point,

$$\lambda \approx 10^{-7} \text{ cm}$$

$$\tau \approx 10^{-11} \text{ s}$$

For H_2 gas in interstellar space, where the density is about 1 molecule/cm³,

$$\lambda \approx 10^{15} \text{ cm}$$

The diameter of H_2 has been taken to be about 1 Å.

From these qualitative estimates, it is expected that in H_2 gas under normal conditions, for example, any nonuniformity in density or temperature over distances of order 10^{-7} cm will be ironed out in the order of 10^{-11} s. Variations in density or temperature over macroscopic distances may persist for a long time.

5.2 EFFUSION

An important quantity governing the behavior of a gas is the ratio of the mean free path to some other characteristic length, such as

The size of the box containing the gas.

The diameter of a hole through which gas molecules may pass.

The wavelength of density fluctuations.

When the mean free path is large compared to any other length in the problem, the gas is said to be in the *collisionless regime*. A practical example is the process of *effusion*, whereby a gas leaks through a very small hole of diameter much smaller than the mean free path—a phenomenon of great interest to all experimentalists who maintain vacuum systems.

In effusion the gas molecules do not collide as they go through the hole. Therefore the flux I through the hole, defined as the number of molecules crossing the hole per second per unit area of the hole, is just the flux of molecules impinging on the surface area of the hole. The contribution to the flux from molecules of velocity \mathbf{v} is given by

$$dI = d^3p v_x f(\mathbf{p})$$

where the x axis is chosen normal to the hole. The total flux is therefore

$$I = \int_{v_x > 0} d^3p v_x f(\mathbf{p})$$

Assuming the Maxwell-Boltzmann distribution, we have

$$\begin{aligned} I &= \frac{nm^3}{(2\pi mkT)^{3/2}} \int_0^\infty dv_x v_x e^{-mv_x^2/2kT} \int_{-\infty}^\infty dv_y e^{-mv_y^2/2kT} \int_{-\infty}^\infty dv_z e^{-mv_z^2/2kT} \\ &= n \sqrt{\frac{kT}{2\pi m}} = \frac{n\bar{v}}{2\sqrt{\pi}} \end{aligned}$$

Eliminating n through $P = nkT$, we obtain

$$I = \frac{P}{\sqrt{2\pi mkT}} \quad (5.6)$$

The inverse proportionality to \sqrt{m} makes the process useful as a means of separating isotopes.

The opposite of the collisionless regime is one in which the mean free path is much smaller than the other characteristic lengths of the problem, exemplified by the flow of a gas through a very large hole. In this case the gas molecules will undergo many collisions as they pass through the hole, and will "thermalize" locally. The prevailing condition is known as the *hydrodynamic regime*, and will be the subject of the rest of this chapter.

5.3 THE CONSERVATION LAWS

To investigate nonequilibrium phenomena, we must solve the Boltzmann transport equation, with given initial conditions, to obtain the distribution function as a function of time. Some rigorous properties of any solution to the Boltzmann equation may be obtained from the fact that in any molecular collision there are dynamical quantities that are rigorously conserved.

Let $\chi(\mathbf{r}, \mathbf{p})$ be any quantity associated with a molecule of velocity \mathbf{p} located at \mathbf{r} , such that in any collision $\{\mathbf{p}_1, \mathbf{p}_2\} \rightarrow \{\mathbf{p}'_1, \mathbf{p}'_2\}$ taking place at \mathbf{r} , we have

$$\chi_1 + \chi_2 = \chi'_1 + \chi'_2 \quad (5.7)$$

where $\chi_1 = \chi(\mathbf{r}_1, \mathbf{p}_1)$, etc. We call χ a conserved property. The following theorem holds.

THEOREM

$$\int d^3p \chi(\mathbf{r}, \mathbf{p}) \left[\frac{\partial f(\mathbf{r}, \mathbf{p}, t)}{\partial t} \right]_{\text{coll}} = 0 \quad (5.8)$$

where $(\partial f / \partial t)_{\text{coll}}$ is the right side of (3.36).*

Proof By definition of $(\partial f / \partial t)_{\text{coll}}$ we have

$$\int d^3p \chi \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = \int d^3p_1 d^3p_2 d^3p'_1 d^3p'_2 \delta^4(P_t - P_1) |T_{fi}|^2 (f'_2 f'_1 - f_2 f_1) \chi_1 \quad (5.9)$$

Making use of the properties of T_{fi} discussed in Section 3.2, and proceeding in a manner similar to the proof of the H theorem, we make each of the following interchanges of integration variables.

First: $\mathbf{p}_1 \rightleftharpoons \mathbf{p}_2$

Next: $\mathbf{p}_1 \rightleftharpoons \mathbf{p}'_1$ and $\mathbf{p}_2 \rightleftharpoons \mathbf{p}'_2$

Next: $\mathbf{p}_1 \rightleftharpoons \mathbf{p}'_2$ and $\mathbf{p}_2 \rightleftharpoons \mathbf{p}'_1$

For each case we obtain a different form for the same integral. Adding the three

*Note that it is not required that f be a solution of the Boltzmann transport equation.

new formulas so obtained to (5.9) and dividing the result by 4 we get

$$\int d^3p \chi \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = \frac{1}{4} \int d^3p_1 d^3p_2 d^3p'_1 d^3p'_2 \delta^4(P_t - P_1) |T_{fi}|^2 \times (f'_2 f'_1 - f_2 f_1) (\chi_1 + \chi_2 - \chi'_1 - \chi'_2) \equiv 0 \quad \blacksquare$$

The conservation theorem relevant to the Boltzmann transport equation is obtained by multiplying the Boltzmann transport equation on both sides by χ and then integrating over \mathbf{p} . The collision term vanishes by virtue of (5.8), and we have*

$$\int d^3p \chi(\mathbf{r}, \mathbf{p}) \left(\frac{\partial}{\partial t} + \frac{p_i}{m} \frac{\partial}{\partial x_i} + F_i \frac{\partial}{\partial p_i} \right) f(\mathbf{r}, \mathbf{p}, t) = 0 \quad (5.10)$$

We may rewrite (5.10) in the form

$$\begin{aligned} \frac{\partial}{\partial t} \int d^3p \chi f + \frac{\partial}{\partial x_i} \int d^3p \chi \frac{p_i}{m} f - \int d^3p \frac{\partial \chi}{\partial x_i} \frac{p_i}{m} f + \int d^3p \frac{\partial}{\partial p_i} (\chi F_i f) \\ - \int d^3p \frac{\partial \chi}{\partial p_i} F_i f - \int d^3p \chi \frac{\partial F_i}{\partial p_i} f = 0 \quad - \int \frac{f}{m} \chi \frac{\partial p_i}{\partial x_i} \quad (5.11) \end{aligned}$$

The fourth term vanishes if $f(\mathbf{r}, \mathbf{p}, t)$ is assumed to vanish when $|\mathbf{p}| \rightarrow \infty$. This conservation theorem is most useful in hydrodynamics, where the velocity $\mathbf{v} = \mathbf{p}/m$ rather than the momentum \mathbf{p} is a directly measurable quantity. Accordingly, we shall reexpress \mathbf{p} in terms of \mathbf{v} , where convenient. We also define the average value $\langle A \rangle$ by

$$\langle A \rangle \equiv \frac{\int d^3p A f}{\int d^3p f} = \frac{1}{n} \int d^3p A f \quad (5.12)$$

where

$$n(\mathbf{r}, t) \equiv \int d^3p f(\mathbf{r}, \mathbf{p}, t) \quad (5.13)$$

We obtain finally the desired theorem:

CONSERVATION THEOREM

$$\frac{\partial}{\partial t} \langle n\chi \rangle + \frac{\partial}{\partial x_i} \langle n v_i \chi \rangle - n \left\langle v_i \frac{\partial \chi}{\partial x_i} \right\rangle - \frac{n}{m} \left\langle F_i \frac{\partial \chi}{\partial v_i} \right\rangle - \frac{n}{m} \left\langle \frac{\partial F_i}{\partial v_i} \chi \right\rangle = 0 \quad (5.14)$$

where χ is any conserved property. Note that $\langle nA \rangle = n \langle A \rangle$ because n is

*The summation convention, whereby a repeated vector index is understood to be summed from 1 to 3, is used.

independent of \mathbf{v} . From now on we restrict our attention to velocity-independent external forces so that the last term of (5.14) may be dropped.

For simple molecules the independent conserved properties are mass, momentum, and energy. For charged molecules we also include the charge, but this extension is trivial. Accordingly we set successively

$$\begin{aligned}\chi &= m && \text{(mass)} \\ \chi &= mv_i \quad (i = 1, 2, 3) && \text{(momentum)} \\ \chi &= \frac{1}{2}m|\mathbf{v} - \mathbf{u}(\mathbf{r}, t)|^2 && \text{(thermal energy)}\end{aligned}$$

where

$$\mathbf{u}(\mathbf{r}, t) \equiv \langle \mathbf{v} \rangle$$

We should then have three independent conservation theorems.

For $\chi = m$ we have immediately

$$\frac{\partial}{\partial t} \langle mn \rangle + \frac{\partial}{\partial x_i} \langle mnv_i \rangle = 0$$

or, introducing the mass density

$$\rho(\mathbf{r}, t) \equiv mn(\mathbf{r}, t)$$

we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (5.15)$$

Next we put $\chi = mv_i$, obtaining

$$\frac{\partial}{\partial t} \langle \rho v_i \rangle + \frac{\partial}{\partial x_j} \langle \rho v_i v_j \rangle - \frac{1}{m} \rho F_i = 0 \quad (5.16)$$

To reduce this further let us write

$$\begin{aligned}\langle v_i v_j \rangle &= \langle (v_i - u_i)(v_j - u_j) \rangle + \langle v_i \rangle u_j + u_i \langle v_j \rangle - u_i u_j \\ &= \langle (v_i - u_i)(v_j - u_j) \rangle + u_i u_j\end{aligned}$$

Substituting this into (5.16) we obtain

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{1}{m} \rho F_i - \frac{\partial}{\partial x_j} \langle \rho (v_i - u_i)(v_j - u_j) \rangle \quad (5.17)$$

Introducing the abbreviation

$$P_{ij} \equiv \rho \langle (v_i - u_i)(v_j - u_j) \rangle$$

which is called the *pressure tensor*, we finally have

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) u_i = \frac{1}{m} F_i - \frac{1}{\rho} \frac{\partial}{\partial x_j} P_{ij} \quad (5.18)$$

Finally we set $\chi = \frac{1}{2}m|\mathbf{v} - \mathbf{u}|^2$. Then

$$\frac{1}{2} \frac{\partial}{\partial t} \langle \rho |\mathbf{v} - \mathbf{u}|^2 \rangle + \frac{1}{2} \frac{\partial}{\partial x_i} \langle \rho v_i |\mathbf{v} - \mathbf{u}|^2 \rangle - \frac{1}{2} \rho \left\langle v_i \frac{\partial}{\partial x_i} |\mathbf{v} - \mathbf{u}|^2 \right\rangle = 0 \quad (5.19)$$

We define the *temperature* by

$$kT \equiv \theta \equiv \frac{1}{3}m \langle |\mathbf{v} - \mathbf{u}|^2 \rangle$$

and the *heat flux* by

$$\mathbf{q} = \frac{1}{2}m\rho \langle (\mathbf{v} - \mathbf{u})|\mathbf{v} - \mathbf{u}|^2 \rangle \neq n\bar{\mathbf{v}}E \text{ in units}$$

We then have

$$\begin{aligned}\frac{1}{2}m\rho \langle v_i |\mathbf{v} - \mathbf{u}|^2 \rangle &= \frac{1}{2}m\rho \langle (v_i - u_i) |\mathbf{v} - \mathbf{u}|^2 \rangle + \frac{1}{2}m\rho u_i \langle |\mathbf{v} - \mathbf{u}|^2 \rangle \\ &= q_i + \frac{3}{2}\rho\theta u_i\end{aligned}$$

and

$$\rho \langle v_i (v_j - u_j) \rangle = \rho \langle (v_i - u_i)(v_j - u_j) \rangle + \rho u_i \langle v_j - u_j \rangle = P_{ij}$$

Thus (5.19) can be written

$$\frac{3}{2} \frac{\partial}{\partial t} (\rho\theta) + \frac{\partial q_i}{\partial x_i} + \frac{3}{2} \frac{\partial}{\partial x_i} (\rho\theta u_i) + mP_{ij} \frac{\partial u_j}{\partial x_i} = 0$$

Since $P_{ij} = P_{ji}$

$$mP_{ij} \frac{\partial u_j}{\partial x_i} = P_{ij} \frac{m}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \equiv P_{ij} \Lambda_{ij}$$

The final form is then obtained after a few straightforward steps:

$$\rho \left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \right) \theta + \frac{2}{3} \frac{\partial}{\partial x_i} q_i = -\frac{2}{3} \Lambda_{ij} P_{ij} \quad (5.20)$$

The three conservation theorems are summarized in (5.21), (5.22), and (5.23).

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{(conservation of mass)} \quad (5.21)$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\rho}{m} \mathbf{F} - \nabla \cdot \vec{\mathbf{P}} \quad \text{(conservation of momentum)} \quad (5.22)$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = -\frac{2}{3} \nabla \cdot \mathbf{q} - \frac{2}{3} \vec{\mathbf{P}} \cdot \vec{\mathbf{\Lambda}} \quad \text{(conservation of energy)} \quad (5.23)$$

where $\vec{\mathbf{P}}$ is a dyadic whose components are P_{ij} , $\nabla \cdot \vec{\mathbf{P}}$ is a vector whose i th component is $\partial P_{ij} / \partial x_j$, and $\vec{\mathbf{P}} \cdot \vec{\mathbf{\Lambda}}$ is a scalar $P_{ij} \Lambda_{ij}$. The auxiliary quantities are

defined as follows.

$$\rho(\mathbf{r}, t) \equiv m \int d^3v f(\mathbf{r}, \mathbf{v}, t) \quad (\text{mass density}) \quad (5.24)$$

$$\mathbf{u}(\mathbf{r}, t) \equiv \langle \mathbf{v} \rangle \quad (\text{average velocity}) \quad (5.25)$$

$$\theta(\mathbf{r}, t) \equiv \frac{1}{3} m \langle |\mathbf{v} - \mathbf{u}|^2 \rangle \quad (\text{temperature}) \quad (5.26)$$

$$\mathbf{q}(\mathbf{r}, t) \equiv \frac{1}{2} m \rho \langle (\mathbf{v} - \mathbf{u}) |\mathbf{v} - \mathbf{u}|^2 \rangle \quad (\text{heat flux vector}) \quad (5.27)$$

$$P_{ij} \equiv \rho \langle (v_i - u_i)(v_j - u_j) \rangle \quad (\text{pressure tensor}) \quad (5.28)$$

$$\Lambda_{ij} \equiv \frac{1}{2} m \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5.29)$$

Although the conservation theorems are exact, they have no practical value unless we can actually solve the Boltzmann transport equation and use the distribution function so obtained to evaluate the quantities (5.24)–(5.29). Despite the fact that these quantities have been given rather suggestive names, their physical meaning, if any, can only be ascertained after the distribution function is known. We shall see that when it is known these conservation theorems become the physically meaningful equations of hydrodynamics.

5.4 THE ZERO-ORDER APPROXIMATION

From now on we shall work in the hydrodynamic regime, where the mean free path is small compared to other characteristic lengths. This means that gas molecules make a large number of collisions within a small space. Consequently they come to local equilibrium rapidly. In the lowest-order approximation it is natural to assume that the gas has a local Maxwell-Boltzmann distribution, with slowly varying temperature, density, and average velocity:

$$f(\mathbf{r}, \mathbf{p}, t) \approx f^{(0)}(\mathbf{r}, \mathbf{p}, t) \quad (5.30)$$

where

$$f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \frac{n}{(2\pi m\theta)^{3/2}} \exp\left[-\frac{m}{2\theta}(\mathbf{v} - \mathbf{u})^2\right] \quad (5.31)$$

where n, θ, \mathbf{u} are all slowly varying functions of \mathbf{r} and t . It is obvious that (5.30) cannot be an exact solution of the Boltzmann transport equation. On the one hand we have

$$\left(\frac{\partial f^{(0)}}{\partial t} \right)_{\text{coll}} = 0 \quad (5.32)$$

because n, θ, \mathbf{u} do not depend on \mathbf{v} . On the other hand it is clear that in general

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \right) f^{(0)}(\mathbf{r}, \mathbf{p}, t) \neq 0 \quad (5.33)$$

We postpone the discussion of the accuracy of the approximation (5.30). For the moment let us assume that it is a good approximation and discuss the physical consequences.

If (5.30) is a good approximation, the left side of (5.33) must be approximately equal to zero. This in turn would mean that n, θ, \mathbf{u} are such that the conservation theorems (5.21)–(5.23) are approximately satisfied. The conservation theorems then become the equations restricting the behaviour of n, θ, \mathbf{u} . To see what they are, we must calculate \mathbf{q} and P_{ij} to the lowest order. The results are denoted respectively by $\mathbf{q}^{(0)}$ and $P_{ij}^{(0)}$. Let $C(\mathbf{r}, t) = n(m/2\pi\theta)^{3/2}$ and $A(\mathbf{r}, t) = m/2\theta$. We easily obtain

$$\begin{aligned} \mathbf{q}^{(0)} &= \frac{1}{2} \frac{m\rho}{n} \int d^3v (\mathbf{v} - \mathbf{u}) |\mathbf{v} - \mathbf{u}|^2 C(\mathbf{r}, t) e^{-A(\mathbf{r}, t)|\mathbf{v} - \mathbf{u}|^2} \\ &= \frac{1}{2} m^2 C(\mathbf{r}, t) \int d^3U \mathbf{U} U^2 e^{-A(\mathbf{r}, t)U^2} = 0 \end{aligned} \quad (5.34)$$

$$\begin{aligned} P_{ij}^{(0)} &= \frac{\rho}{n} C(\mathbf{r}, t) \int d^3v (v_i - u_i)(v_j - u_j) e^{-A(\mathbf{r}, t)|\mathbf{v} - \mathbf{u}|^2} \\ &= mC(\mathbf{r}, t) \int d^3U U_i U_j e^{-A(\mathbf{r}, t)U^2} = \delta_{ij} P \end{aligned} \quad (5.35)$$

where

$$P = \frac{1}{3} \rho \left(\frac{m}{2\pi\theta} \right)^{3/2} \int d^3U \mathbf{U} U^2 e^{-A(\mathbf{r}, t)U^2} = n\theta \quad (5.36)$$

which is the local hydrostatic pressure.

Substituting these into (5.21) and (5.23), and noting that

$$\begin{aligned} \nabla \cdot \vec{\mathbf{P}}^{(0)} &= \nabla P \\ \vec{\mathbf{P}}^{(0)} \cdot \vec{\mathbf{\Lambda}} &= P \sum_{i=1}^3 \Lambda_{ii} = mP \nabla \cdot \mathbf{u} \end{aligned}$$

We obtain the equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{continuity equation}) \quad (5.37)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + \frac{1}{\rho} \nabla P = \frac{\mathbf{F}}{m} \quad (\text{Euler's equation}) \quad (5.38)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta + \frac{1}{c_V} (\nabla \cdot \mathbf{u}) \theta = 0 \quad (\text{Stokes eqn}) \quad (5.39)$$

where $c_V = \frac{5}{2}$. These are the hydrodynamic equations for the nonviscous flow of a gas. They possess solutions describing flow patterns that persist indefinitely. Thus, in this approximation, the local Maxwell-Boltzmann distribution never decays to the true Maxwell-Boltzmann distribution. This is in rough accord with experience, for we know that a hydrodynamic flow, left to itself, takes a long time to die out.

Although derived for dilute gases, (5.37)–(5.39) are also used for liquids because these equations can also be derived through heuristic arguments which indicate that they are of a more general validity.

We shall now briefly point out some of the consequences of (5.37)–(5.39) that are of practical interest.

The quantity $(\partial/\partial t + \mathbf{u} \cdot \nabla) X$ is known as the “material derivative of X ,” because it is the time rate of change of X to an observer moving with the local average velocity \mathbf{u} . Such an observer is said to be moving along a streamline. We now show that in the zero-order approximation a dilute gas undergoes only adiabatic transformations to an observer moving along a streamline. Equations (5.37) and (5.39) may be rewritten as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho &= -\rho \nabla \cdot \mathbf{u} \\ -\frac{3}{2} \frac{\rho}{\theta} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta &= \rho \nabla \cdot \mathbf{u} \end{aligned}$$

Adding these two equations we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho - \frac{3}{2} \frac{\rho}{\theta} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = 0$$

or

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (\rho \theta^{-3/2}) = 0 \quad (5.40)$$

Using the equation of state $P = \rho \theta / m$ we can convert (5.40) to the condition

$$P \rho^{-5/3} = \text{constant} \quad (\text{along a streamline}) \quad (5.41)$$

This is the condition for adiabatic transformation for an ideal gas, since $c_P/c_V = \frac{5}{3}$.

Next we derive the linear equation for a sound wave. Let us restrict ourselves to the case in which \mathbf{u} and all the space and time derivatives of \mathbf{u} , ρ , and θ are small quantities of the first order. For $\mathbf{F} = 0$, (5.37) and (5.38) may be replaced by

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} = 0 \quad (5.42)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla P = 0 \quad (5.43)$$

$$\frac{3}{2} \rho \frac{\partial \theta}{\partial t} - \theta \frac{\partial \rho}{\partial t} = 0 \quad (5.44)$$

where quantities smaller than first-order ones are neglected. Note that (5.44) is none other than (5.40) or (5.41). Taking the divergence of (5.43) and the time derivative of (5.42), and subtracting one resulting equation from the other, we

obtain

$$\nabla^2 P - \frac{\partial^2 \rho}{\partial t^2} = 0 \quad (5.45)$$

in which higher-order quantities are again neglected. Now P is a function of ρ and θ , but the latter are not independent quantities, being related to each other through the condition of adiabatic transformation (5.44). Hence we may regard P as a function of ρ alone, and write

$$\nabla^2 P = \nabla \cdot \left[\left(\frac{\partial P}{\partial \rho} \right)_S \nabla \rho \right] \approx \left(\frac{\partial P}{\partial \rho} \right)_S \nabla^2 \rho$$

where $(\partial P/\partial \rho)_S$ is the adiabatic derivative, related to the adiabatic compressibility κ_S by

$$\kappa_S = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P} \right)_S = \frac{3}{5} \frac{m}{\rho \theta} \quad (5.46)$$

Thus (5.45) can be written in the form

$$\nabla^2 \rho - \rho \kappa_S \frac{\partial^2 \rho}{\partial t^2} = 0 \quad (5.47)$$

which is a wave equation for ρ , describing a sound wave with a velocity of propagation c given by

$$c = \frac{1}{\sqrt{\rho \kappa_S}} = \sqrt{\frac{5}{3} \frac{\theta}{m}} = \sqrt{\frac{5}{6}} \bar{v} \quad (5.48)$$

It is hardly surprising that the adiabatic compressibility enters here, because in the present approximation there can be no heat conduction in the gas, as (5.34) indicates.

Finally consider the case of steady flow under the influence of a conservative external force field, i.e., under the conditions

$$\begin{aligned} \mathbf{F} &= -\nabla \phi \\ \frac{\partial \mathbf{u}}{\partial t} &= 0 \end{aligned} \quad (5.49)$$

Using the vector identity

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla (u^2) - \mathbf{u} \times (\nabla \times \mathbf{u}) \quad (5.50)$$

we can rewrite (5.38) as follows

$$\nabla \left(\frac{1}{2} u^2 + \frac{1}{\rho} P + \frac{1}{m} \phi \right) = \mathbf{u} \times (\nabla \times \mathbf{u}) - \frac{\theta}{m} \frac{\nabla \rho}{\rho} \quad (5.51)$$

Two further specializations are of interest. First, in the case of uniform density

and irrotational flow, namely, $\nabla\rho = 0$ and $\nabla \times \mathbf{u} = 0$, we have

$$\nabla \left(\frac{1}{2}u^2 + \frac{1}{\rho}P + \frac{1}{m}\phi \right) = 0 \quad (5.52)$$

which is *Bernoulli's equation*. Second, in the case of uniform temperature and irrotational flow, namely, $\nabla\theta = 0$ and $\nabla \times \mathbf{u} = 0$, we have

$$\nabla \left(\frac{1}{2}u^2 + \frac{1}{m}\phi \right) = -\frac{\theta}{m}\nabla(\log\rho)$$

which may be immediately integrated to yield

$$\rho = \rho_0 \exp \left[-\frac{1}{\theta} \left(\frac{1}{2}mu^2 + \phi \right) \right] \quad (5.53)$$

where ρ_0 is an arbitrary constant.

5.5 THE FIRST-ORDER APPROXIMATION

We now give an estimate of the error incurred in the zero-order approximation (5.30). Let $f(\mathbf{r}, \mathbf{p}, t)$ be the exact distribution function, and let

$$g(\mathbf{r}, \mathbf{p}, t) \equiv f(\mathbf{r}, \mathbf{p}, t) - f^{(0)}(\mathbf{r}, \mathbf{p}, t) \quad (5.54)$$

We are interested in the magnitude of g as compared to $f^{(0)}$. First let us estimate the order of magnitude of $(\partial f / \partial t)_{\text{coll}}$. We have, by definition,

$$\begin{aligned} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} &= \int d^3p_2 d^3p_1' d^3p_2' \delta^4(P_t - P_i) |T_{fi}|^2 (f_2' f_1' - f_2 f_1) \\ &\approx \int d^3p_2 d^3p_1' d^3p_2' \delta^4(P_t - P_i) |T_{fi}|^2 \\ &\quad \times (f_2^{(0)'} g_1' - f_2^{(0)} g_1 + g_2' f_1^{(0)'} - g_2 f_1^{(0)}) \end{aligned} \quad (5.55)$$

where we have used (5.54), the fact that $(\partial f^{(0)} / \partial t)_{\text{coll}} = 0$, and the assumption that g is a small quantity whose square can be neglected. An order-of-magnitude estimate of (5.55) may be obtained by calculating the second term of the right side of (5.55), which is

$$-g(\mathbf{r}, \mathbf{p}_1, t) \int d^3p_2 \sigma_{\text{tot}} |\mathbf{v}_2 - \mathbf{v}_1| f_2^{(0)} = -\frac{g(\mathbf{r}, \mathbf{p}_1, t)}{\tau} \quad (5.56)$$

where τ is a number of the order of magnitude of the collision time. Thus if we put

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \approx -\frac{f - f^{(0)}}{\tau} \quad (5.57)$$

we obtain results that are qualitatively correct.* With (5.57) the Boltzmann transport equation becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \right) (f^{(0)} + g) \approx -\frac{g}{\tau} \quad (5.58)$$

Assuming $g \ll f^{(0)}$, we can neglect g on the left side of (5.58). Assume further that $f^{(0)}$ varies by a significant amount (i.e., of the order of itself) only when $|\mathbf{r}|$ varies by a distance L . Then (5.58) furnishes the estimate

$$\begin{aligned} \frac{f^{(0)}}{L} &\approx -\frac{g}{\tau} \\ \text{or} \\ \frac{g}{f^{(0)}} &\approx -\frac{\lambda}{L} \end{aligned} \quad (5.59)$$

where λ is a length of the order of the mean free path. From these considerations we conclude that $f^{(0)}$ is a good approximation if the local density, temperature, and velocity have characteristic wavelengths L much larger than the mean free path λ . The corrections to $f^{(0)}$ would be of the order of λ/L .

A systematic expansion of f in powers of λ/L is furnished by the Chapman-Enskog expansion, which is somewhat complicated. In order not to lose sight of the physical aspects of the problem, we give a qualitative discussion of the first-order approximation based on the approximate equation (5.58). The precise value of τ cannot be ascertained. For the present we have to be content with the knowledge that τ is of the order of the collision time. Thus we put

$$f = f^{(0)} + g \quad (5.60)$$

where, with (5.58), we take

$$g = -\tau \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \right) f^{(0)} \quad (5.61)$$

To calculate g , note that $f^{(0)}$ depends on \mathbf{r} and t only through the functions ρ , θ , and \mathbf{u} . Thus we need the derivatives

$$\begin{aligned} \frac{\partial f^{(0)}}{\partial \rho} &= \frac{f^{(0)}}{\rho} \\ \frac{\partial f^{(0)}}{\partial \theta} &= \frac{1}{\theta} \left(\frac{m}{2\theta} U^2 - \frac{3}{2} \right) f^{(0)} \\ \frac{\partial f^{(0)}}{\partial u_i} &= \frac{m}{\theta} U_i f^{(0)} \\ \frac{\partial f^{(0)}}{\partial v_i} &= -\frac{m}{\theta} U_i f^{(0)} \end{aligned} \quad (5.62)$$

*Techniques useful for solving the Boltzmann transport equation, together with results for a few simple intermolecular potentials, may be found in S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 2nd ed. (Cambridge University Press, Cambridge, 1952).

where

$$\mathbf{U} \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t) \quad (5.63)$$

Hence

$$\begin{aligned} g &= -\tau \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \frac{F_i}{m} \frac{\partial}{\partial v_i} \right) f^{(0)} \\ &= -\tau f^{(0)} \left[\frac{1}{\rho} D(\rho) + \frac{1}{\theta} \left(\frac{m}{2\theta} U^2 - \frac{3}{2} \right) D(\theta) + \frac{m}{\theta} U_j D(u_j) - \frac{1}{\theta} \mathbf{F} \cdot \mathbf{U} \right] \end{aligned} \quad (5.64)$$

where

$$D(X) \equiv \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) X \quad (5.65)$$

Using the zero-order hydrodynamic equations (5.37)–(5.39), we can show that

$$\begin{aligned} D(\rho) &= -\rho(\nabla \cdot \mathbf{u}) + \mathbf{U} \cdot \nabla \rho \\ D(\theta) &= -\frac{2}{3} \theta \nabla \cdot \mathbf{u} + \mathbf{U} \cdot \nabla \theta \\ D(u_j) &= -\frac{1}{\rho} \frac{\partial P}{\partial x_j} + \frac{F_j}{m} + U_i \frac{\partial u_j}{\partial x_i} \end{aligned} \quad (5.66)$$

where $P = \rho\theta/m$. Substituting these into (5.64) we obtain

$$\begin{aligned} g &= -\tau f^{(0)} \left[-(\nabla \cdot \mathbf{u}) + \mathbf{U} \cdot \frac{\nabla \rho}{\rho} + \frac{1}{\theta} \left(\frac{m}{2\theta} U^2 - \frac{3}{2} \right) \left(-\frac{2}{3} \theta \nabla \cdot \mathbf{u} + \mathbf{U} \cdot \nabla \theta \right) \right. \\ &\quad \left. + \frac{m}{\theta} \left(-\mathbf{U} \cdot \frac{\nabla P}{\rho} + \mathbf{U} \cdot \frac{\mathbf{F}}{m} + U_i U_j \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{\theta} \mathbf{F} \cdot \mathbf{U} \right] \end{aligned} \quad (5.67)$$

which, after some rearrangement and cancellation of terms, becomes

$$g = -\tau \left[\frac{1}{\theta} \frac{\partial \theta}{\partial x_i} U_i \left(\frac{m}{2\theta} U^2 - \frac{5}{2} \right) + \frac{1}{\theta} \Lambda_{ij} (U_i U_j - \frac{1}{3} \delta_{ij} U^2) \right] f^{(0)} \quad (5.67)$$

where Λ_{ij} is defined by (5.29).

It is now necessary to calculate \mathbf{q} and P_{ij} with the help of (5.60) to obtain the equations of hydrodynamics to the first order. We have

$$\mathbf{q} = \frac{m\rho}{2n} \int d^3p (\mathbf{v} - \mathbf{u}) |\mathbf{v} - \mathbf{u}|^2 g$$

Noting that the second term of (5.67) does not contribute to this integral, we obtain

$$\mathbf{q} = -\frac{\tau m^5}{2} \int d^3U \mathbf{U} U^2 \left(\frac{m}{2\theta} U^2 - \frac{5}{2} \right) \frac{1}{\theta} U_i \frac{\partial \theta}{\partial x_i} f^{(0)}$$

or

$$\mathbf{q} = -K \nabla \theta \quad (5.68)$$

where

$$K = \frac{m^5 \tau}{6\theta} \int d^3U U U^4 \left(\frac{m}{2\theta} U^2 - \frac{5}{2} \right) f^{(0)} = \frac{5}{2} \tau n \quad (5.69)$$

It is clear from (5.68) that K is to be identified as the coefficient of thermal conductivity. It is also clear that $|\mathbf{q}|$ is a small quantity of the first order, being of the order of λ/L .

For the pressure tensor P_{ij} , only the second term of (5.67) contributes:

$$P_{ij} = \frac{\rho}{n} \int d^3p (v_i - u_i)(v_j - u_j)(f^{(0)} + g) = \delta_{ij} P + P'_{ij} \quad (5.70)$$

where $P = \rho\theta/m$ and

$$P'_{ij} = -\frac{\tau \rho m^3}{\theta n} \Lambda_{kl} \int d^3U U_i U_j (U_k U_l - \frac{1}{3} \delta_{kl} U^2) f^{(0)} \quad (5.71)$$

To evaluate this, note that P'_{ij} is a symmetric tensor of zero trace (i.e., $\sum_{i=1}^3 P'_{ii} = 0$), and it depends linearly on the symmetric tensor Λ_{ij} . Therefore P'_{ij} must have the form

$$P'_{ij} = -\frac{2\mu}{m} \left(\Lambda_{ij} - \frac{m}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right) \quad (5.72)$$

where $m\nabla \cdot \mathbf{u}$ is none other than the trace of Λ_{ij} :

$$\sum_{i=1}^3 \Lambda_{ii} = m \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = m\nabla \cdot \mathbf{u} \quad (5.73)$$

and μ is a constant. It remains to calculate μ . For this purpose it suffices to calculate any component of P'_{ij} , e.g., P'_{12} . From (5.71) we have

$$\begin{aligned} P'_{12} &= -\frac{\tau m^4}{\theta} \Lambda_{kl} \int d^3U U_1 U_2 (U_k U_l - \frac{1}{3} \delta_{kl} U^2) f^{(0)} \\ &= -2 \frac{\tau m^4}{\theta} \Lambda_{12} \int d^3U U_1^2 U_2^2 f^{(0)} \end{aligned}$$

Therefore

$$\mu = \frac{\tau m^5}{\theta} \int d^3U U_1^2 U_2^2 f^{(0)} = \tau n \theta \quad (5.74)$$

With this we have

$$P_{ij} = \delta_{ij} P - \frac{2\mu}{m} \left(\Lambda_{ij} - \frac{m}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right) \quad (5.75)$$

The second term is of the order of λ/L . The coefficient μ turns out to be the coefficient of viscosity, as we show shortly.

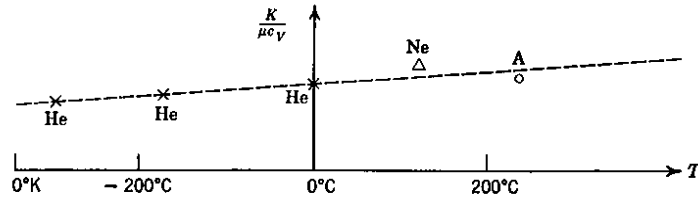


Fig. 5.1 Ratio of thermal conductivity to the product of viscosity and specific heat for different dilute gases.

A comparison of (5.74) with (5.69) shows that

$$\frac{K}{\mu} = \frac{5}{2} = \frac{5}{3}c_v \quad (5.76)$$

Since the unknown collision time τ drops out in this relation, we might expect (5.76) to be of quantitative significance. A plot of some experimental data for different dilute gases in Fig. 5.1 shows that it is indeed so.

Let us put, with (5.6),

$$\tau \approx \sqrt{\frac{m}{kT}} \frac{1}{na^2} \quad (5.77)$$

where a is the molecular diameter. Then we find that

$$\mu \approx K \approx \frac{\sqrt{mkT}}{a^2} \quad (5.78)$$

5.6 VISCOSITY

To show that (5.74) is the coefficient of viscosity, we independently calculate the coefficient of viscosity using its experimental definition. Consider a gas of uniform and constant density and temperature, with an average velocity given by

$$\begin{aligned} u_x &= A + By \\ u_y &= u_z = 0 \end{aligned} \quad (5.79)$$

where A and B are constants. The gas may be thought of as being composed of different layers sliding over each other, as shown in Fig. 5.2. Draw any plane perpendicular to the y axis, as shown by the dotted line in Fig. 5.2. Let F' be the frictional force experienced by the gas above this plane, per unit area of the plane. Then the coefficient of viscosity μ is experimentally defined by the relation

$$F' = -\mu \frac{\partial u_x}{\partial y} \quad (5.80)$$

The gas above the plane experiences a frictional force by virtue of the fact that it

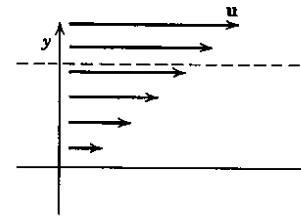


Fig. 5.2 Horizontal flow of a gas with average velocity increasing linearly with height.

suffers a net loss of "x component of momentum" to the gas below. Thus

$$F' \equiv \text{net amount of "x component of momentum" transported per sec across unit area in the y direction} \quad (5.81)$$

The quantity being transported is $m(v_x - u_x)$, whereas the flux effective in the transport is $n(v_y - u_y)$. Hence we have

$$F' = mn \langle (v_x - u_x)(v_y - u_y) \rangle = m^4 \int d^3v (v_x - u_x)(v_y - u_y)(f^{(0)} + g) \quad (5.82)$$

We easily see that the term $f^{(0)}$ does not contribute to the integral in (5.82). The first correction g may be obtained directly from the approximate Boltzmann transport equation

$$\begin{aligned} \mathbf{v} \cdot \nabla f^{(0)} &= -\frac{g}{\tau} \\ g &= -\frac{\tau m}{\theta} v_y (v_x - u_x) B f^{(0)} = -\frac{\tau m}{\theta} U_y U_x \frac{\partial u_x}{\partial y} f^{(0)} \end{aligned} \quad (5.83)$$

where $\mathbf{U} \equiv \mathbf{v} - \mathbf{u}$. Thus

$$F' = -\frac{\partial u_x}{\partial y} \frac{\tau m^5}{\theta} \int d^3U U_x^2 U_y^2 f^{(0)} \quad (5.84)$$

A comparison between this and (5.80) yields

$$\mu = \frac{\tau m^5}{\theta} \int d^3U U_x^2 U_y^2 f^{(0)} \quad (5.85)$$

which is identical with (5.74).

From the nature of this derivation it is possible to understand physically why μ has the order of magnitude given by (5.78). Across the imaginary plane mentioned previously, a net transport of momentum exists, because molecules constantly cross this plane in both directions. The flux is the same in both directions, being of the order of $n\sqrt{kT/m}$. On the average, however, those that cross from above to below carry more "x component of momentum" than the opposite ones, because the average velocity u_x is greater above than below. Since

most molecules that cross the plane from above originated within a mean free path λ above the plane, their u_x is in excess of the local u_x below the plane by the amount $\lambda(\partial u_x/\partial y)$. Hence the net amount of "x component of momentum" transported per second from above to below, per unit area of the plane, is

$$\lambda nm \sqrt{\frac{kT}{m}} \frac{\partial u_x}{\partial y} = \frac{\sqrt{mkT}}{a^2} \frac{\partial u_x}{\partial y} \quad (5.86)$$

Therefore

$$\mu \approx \frac{\sqrt{mkT}}{a^2} \quad (5.87)$$

It is interesting to note that according to (5.87) μ is independent of the density for a given temperature. When Maxwell first derived this fact, he was so surprised that he put it to experimental test by observing the rate of damping of a pendulum suspended in gases of different densities. To his satisfaction, it was verified.

According to (5.87) the coefficient of viscosity increases as the molecular diameter decreases, everything else being constant. This is physically easy to understand because the mean free path λ increases with decreasing molecular diameter. For a given gradient $\partial u_x/\partial y$, the momentum transported across any plane normal to the y axis obviously increases as λ increases. When λ becomes so large that it is comparable to the size of the container of the gas, the whole method adopted here breaks down, and the coefficient of viscosity ceases to be a meaningful concept.

As a topic related to the concept of viscosity we consider the boundary condition for a gas flowing past a wall. A gas, unlike a liquid, does not stick to the wall of its container. Rather, it slips by with an average velocity u_0 . To determine u_0 , it is necessary to know how the gas molecules interact with the wall. We make the simplifying assumption that a fraction $1 - \alpha$ of the molecules striking the wall is reflected elastically while the remaining fraction α is absorbed by the wall, only to return to the gas later with thermal velocity. The number α is called the *coefficient of accommodation*. Suppose the wall is the xy plane, as shown in Fig. 5.3. Then the downward flux of particles is given by

$$m^3 \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_0^{\infty} dv_z n v_z f^{(0)} = n \sqrt{\frac{\theta}{2\pi m}} \quad (5.88)$$

The particles that reach the wall came from a mean free path λ above the wall.

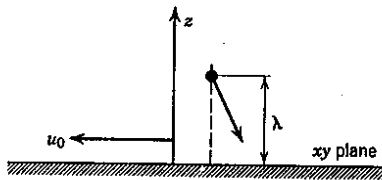


Fig. 5.3 A gas slipping past a wall.

Thus the gas loses to a unit area of the wall an amount of momentum per second equal to

$$F' = -\alpha nm \sqrt{\frac{\theta}{2\pi m}} \left[u_0 + \lambda \left(\frac{\partial u}{\partial z} \right)_0 \right] \quad (5.89)$$

where $(\partial u/\partial z)_0$ is the normal gradient of u at the wall. This is the force of friction per unit area that the wall exerts on the gas, and must equal $-\mu(\partial u/\partial z)_0$. Hence the boundary condition at the wall is

$$\alpha nm \sqrt{\frac{\theta}{2\pi m}} \left[u_0 + \lambda \left(\frac{\partial u}{\partial z} \right)_0 \right] = \mu \left(\frac{\partial u}{\partial z} \right)_0$$

or

$$u_0 = \left(\sqrt{\frac{2\pi}{m\theta}} \frac{\mu}{n\alpha} - \lambda \right) \left(\frac{\partial u}{\partial z} \right)_0 \quad (5.90)$$

Using $\mu = \tau n \theta$ and $\lambda = \beta \tau \sqrt{2\pi\theta/m}$, where β is a constant of the order of unity, we obtain the boundary condition

$$u_0 = s \lambda \left(\frac{\partial u}{\partial z} \right)_0 \quad (5.91)$$

where

$$s = \frac{1 - \alpha\beta}{\alpha\beta}$$

is an empirical constant which may be called the "slipping coefficient." When $s = 0$ there is no slipping at the wall. In general the velocity of slip is equal to the velocity in the gas at a distance of s mean free paths from the wall. Usually $s\lambda$ is a few mean free paths.

5.7 VISCOUS HYDRODYNAMICS

The equations of hydrodynamics in the first-order approximation can be obtained by substituting \mathbf{q} and P_{ij} , given respectively in (5.68) and (5.75), into the conservation theorems (5.21)–(5.23). We first evaluate a few relevant quantities.

$$\nabla \cdot \mathbf{q} = -\nabla \cdot (K \nabla \theta) = -K \nabla^2 \theta - \nabla K \cdot \nabla \theta \quad (5.92)$$

$$\frac{\partial P_{ij}}{\partial x_j} = \frac{\partial P}{\partial x_i} - \mu \left[\nabla^2 u_i + \frac{1}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{u}) \right] - \frac{2}{m} \frac{\partial \mu}{\partial x_j} \left(\Lambda_{ij} - \frac{m}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right) \quad (5.93)$$

$$P_{ij} \Lambda_{ij} = mP (\nabla \cdot \mathbf{u}) - \frac{2\mu}{m} \Lambda_{ij} \Lambda_{ij} + \frac{2}{3} \mu m (\nabla \cdot \mathbf{u})^2 \quad (5.94)$$

The quantity $\Lambda_{ij}\Lambda_{ij}$ can be further reduced:

$$\Lambda_{ij}\Lambda_{ij} = \frac{m^2}{4} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{m^2}{2} \frac{\partial u_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Now we reduce the two terms above separately:

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left(u_i \frac{\partial u_i}{\partial x_j} \right) - u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{1}{2} \nabla^2 (u^2) - \mathbf{u} \cdot \nabla^2 \mathbf{u}$$

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} &= \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \\ &= -2(\nabla \times \mathbf{u})^2 + 2 \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} \end{aligned}$$

Hence

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = -(\nabla \times \mathbf{u})^2 + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}$$

and finally

$$\Lambda_{ij}\Lambda_{ij} = \frac{m^2}{2} \left[\nabla^2 (u^2) - 2\mathbf{u} \cdot \nabla^2 \mathbf{u} - |\nabla \times \mathbf{u}|^2 \right] \quad (5.95)$$

Substituting (5.92)–(5.94) into (5.21)–(5.23) we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (5.96)$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\mathbf{F}}{m} - \nabla \left(P - \frac{\mu}{3} \nabla \cdot \mathbf{u} \right) + \mu \nabla^2 \mathbf{u} + \mathbf{R} \quad (5.97)$$

$$\begin{aligned} \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta &= \frac{K}{c_V} \nabla^2 \theta + \frac{1}{c_V} \nabla K \cdot \nabla \theta - \frac{1}{c_V} \\ &\times \left[m\rho(\nabla \cdot \mathbf{u}) + \frac{2}{3}\mu m(\nabla \cdot \mathbf{u})^2 - \mu m \{ \nabla^2 (u^2) - 2\mathbf{u} \cdot \nabla^2 \mathbf{u} - |\nabla \times \mathbf{u}|^2 \} \right] \end{aligned} \quad (5.98)$$

where $c_V = \frac{3}{2}$ and \mathbf{R} is a vector whose components are given by

$$R_i = \frac{2}{m} \frac{\partial \mu}{\partial x_j} \left(\Lambda_{ij} - \frac{m}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right) \quad (5.99)$$

In these equations the quantities of first-order smallness are μ , K , \mathbf{u} , and the derivatives of ρ , θ , and \mathbf{u} . Keeping only quantities of first-order smallness, we can neglect all terms involving derivatives of μ and K and the last four terms on the right side of (5.98). We then have the equations of hydrodynamics to the first

order:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{continuity equation}) \quad (5.100)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho} \nabla \left(P - \frac{\mu}{3} \nabla \cdot \mathbf{u} \right) + \frac{\mu}{\rho} \nabla^2 \mathbf{u} \quad (\text{Navier-Stokes equation}) \quad (5.101)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = -\frac{1}{c_V} (\nabla \cdot \mathbf{u}) \theta + \frac{K}{\rho c_V} \nabla^2 \theta \quad (\text{heat conduction equation}) \quad (5.102)$$

where $c_V = \frac{3}{2}$. The boundary condition to be used when a wall is present is the slip boundary condition (5.91).

If $\mathbf{u} = 0$, (5.102) reduces to

$$\rho c_V \frac{\partial \theta}{\partial t} - K \nabla^2 \theta = 0 \quad (5.103)$$

which is the familiar diffusion equation governing heat conduction. This equation can be derived intuitively from the fact that $\mathbf{q} = -K \nabla \theta$. Although we have proved this fact only for a dilute gas, it is experimentally correct for liquids and solids as well. For this reason (5.103) is often applied to systems other than a dilute gas.

The Navier-Stokes equation can also be derived on an intuitive basis provided we take the meaning of viscosity from experiments. We discuss this derivation in the next section.

5.8 THE NAVIER-STOKES EQUATION

We give a phenomenological derivation of the Navier-Stokes equation to show why it is expected to be valid even for liquids. Some examples of its use are then discussed.

Consider a small element of fluid whose volume is $dx_1 dx_2 dx_3$ and whose velocity is $\mathbf{u}(\mathbf{r}, t)$. According to Newton's second law the equation of motion of this element of fluid is

$$m \frac{d\mathbf{u}}{dt} = \mathcal{F}$$

where m is the mass of the fluid element and \mathcal{F} is the total force acting on the fluid element. Let the mass density of the fluid be ρ and let there be two forces acting on any element of fluid: A force due to agents external to the fluid, and a force due to neighboring fluid elements. These forces *per unit volume* will be

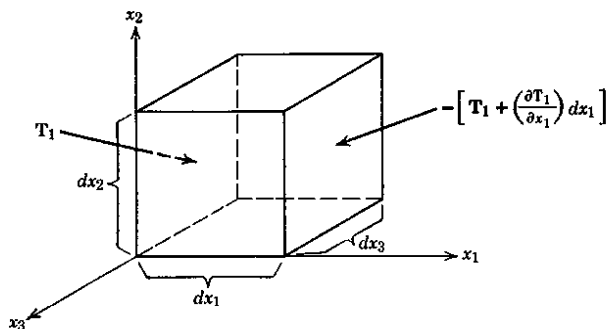


Fig. 5.4 Forces acting on an element of fluid.

respectively denoted by \mathbf{F}_1 and \mathbf{G} . Thus we can write

$$m = \rho dx_1 dx_2 dx_3$$

$$\mathcal{F} = (\mathbf{F}_1 + \mathbf{G}) dx_1 dx_2 dx_3$$

Therefore Newton's second law for a fluid element takes the form

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \mathbf{F}_1 + \mathbf{G} \quad (5.104)$$

Thus the derivation of the Navier-Stokes equation reduces to the derivation of a definite expression for \mathbf{G} .

Let us choose a coordinate system such that the fluid element under consideration is a cube with edges along the three coordinate axes, as shown in Fig. 5.4. The six faces of this cube are subjected to forces exerted by neighboring fluid elements. The force on each face is such that its direction is determined by the direction of the normal vector to the face. That is, its direction depends on which side of the face is considered the "outside." This is physically obvious if we remind ourselves that this force arises from hydrostatic pressure and viscous drag. Let \mathbf{T}_i be the force per unit area acting on the face whose normal lies along the x_i axis. Then the forces *per unit area* acting on the two faces normal to the x_i axis are, respectively (see Fig. 5.4),

$$\mathbf{T}_i, \quad - \left(\mathbf{T}_i + \frac{\partial \mathbf{T}_i}{\partial x_i} dx_i \right) \quad (i = 1, 2, 3) \quad (5.105)$$

The total force acting on the cube by neighboring fluid elements is then given by

$$\mathbf{G} dx_1 dx_2 dx_3 = - \left(\frac{\partial \mathbf{T}_1}{\partial x_1} + \frac{\partial \mathbf{T}_2}{\partial x_2} + \frac{\partial \mathbf{T}_3}{\partial x_3} \right) dx_1 dx_2 dx_3 \quad (5.106)$$

We denote the components of the vectors $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ as follows:

$$\begin{aligned} \mathbf{T}_1 &= (P_{11}, P_{12}, P_{13}) \\ \mathbf{T}_2 &= (P_{21}, P_{22}, P_{23}) \\ \mathbf{T}_3 &= (P_{31}, P_{32}, P_{33}) \end{aligned} \quad (5.107)$$

Then

$$G_i = - \frac{\partial P_{ji}}{\partial x_j} \quad (5.108)$$

or

$$\mathbf{G} = - \nabla \cdot \vec{\mathbf{P}} \quad (5.109)$$

With this, (5.104) becomes

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \mathbf{F}_1 - \nabla \cdot \vec{\mathbf{P}} \quad (5.110)$$

which is of the same form as (5.22) if we set $\mathbf{F}_1 = \rho \mathbf{F}/m$, where \mathbf{F} is the external force per molecule and m is the mass of a molecule. To derive the Navier-Stokes equation, we only have to deduce a more explicit form for P_{ij} . We postulate that (5.110) is valid, whatever the coordinate system we choose. It follows that P_{ij} is a tensor.

We assume the fluid under consideration to be isotropic, so that there can be no intrinsic distinction among the axes x_1, x_2, x_3 . Accordingly we must have

$$P_{11} = P_{22} = P_{33} \equiv P \quad (5.111)$$

where P is by definition the hydrostatic pressure. Thus P_{ij} can be written in the form

$$P_{ij} = \delta_{ij} P + P'_{ij} \quad (5.112)$$

where P'_{ij} is a traceless tensor, namely,

$$\sum_{i=1}^3 P'_{ii} = 0 \quad (5.113)$$

This follows from the fact that (5.113) is true in one coordinate system and that the trace of a tensor is independent of the coordinate system.

Next we make the physically reasonable assumption that the fluid element under consideration, which is really a point in the fluid, has no intrinsic angular momentum. This assumption implies that P_{ij} , and hence P'_{ij} , is a symmetric tensor:

$$P'_{ij} = P'_{ji} \quad (5.114)$$

To see this we need only remind ourselves of the meaning of, for example, P'_{12} . A glance at Fig. 5.5a makes (5.114) obvious.

Finally we incorporate into P_{ij} the empirical connection between the shear force applied to a fluid element and the rate of deformation of the same fluid element. A shear force F' per unit area acting parallel to a face of a cube of fluid

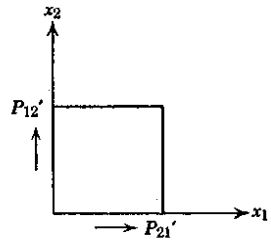


Fig. 5.5a Nonrotation of fluid element implies $P'_{12} = P'_{21}$.

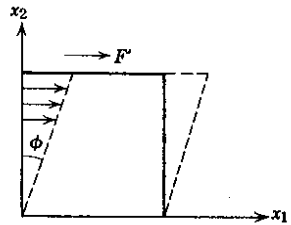


Fig. 5.5b Deformation of fluid element due to shear force.

tends to stretch the cube into a parallelepiped at a rate given by $R' = \mu(d\phi/dt)$, where μ is the coefficient of viscosity and ϕ is the angle shown in Fig. 5.5b.

Consider now the effect of P'_{12} on one fluid element. It can be seen from Fig. 5.5c, where P'_{12} is indicated in its positive sense in accordance with (5.105), that

$$P'_{21} = -\mu \left(\frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} \right) = -\mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \quad (5.115)$$

In general we have

$$P'_{ij} = -\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i \neq j) \quad (5.116)$$

To make P'_{ij} traceless we must take

$$P'_{ij} = -\mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right] \quad (5.117)$$

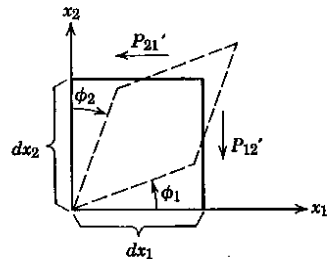


Fig. 5.5c P'_{12} as shear force.

Therefore

$$P_{ij} = \delta_{ij} P - \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right] \quad (5.118)$$

which is identical in form to (5.75). This completes the phenomenological derivation, which makes it plausible that the Navier-Stokes equation is valid for dilute gas and dense liquid alike.

5.9 EXAMPLES IN HYDRODYNAMICS

To illustrate the mathematical techniques of dealing with the equations of hydrodynamics (5.110)–(5.102), we consider two examples of the application of the Navier-Stokes equation to a liquid.

Incompressible Flow

We consider the following problem: A sphere of radius r is moving with instantaneous velocity \mathbf{u}_0 in an infinite, nonviscous, incompressible fluid of constant density in the absence of external force. The Navier-Stokes equation reduces to Euler's equation:

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla P \quad (5.119)$$

where \mathbf{u} is the velocity field of the liquid and P the pressure as given by the equation of state of the fluid. Let us choose the center of the sphere to be the origin of the coordinate system and label any point in space by either the rectangular coordinates (x, y, z) or the spherical coordinates (r, θ, ϕ) . The boundary conditions shall be such that the normal component of \mathbf{u} vanishes on the surface of the sphere and that the liquid is at rest at infinity:

$$\begin{aligned} [\mathbf{r} \cdot \mathbf{u}(\mathbf{r})]_{r=a} - (\mathbf{r} \cdot \mathbf{u}_0)_{r=a} &= 0 \\ \mathbf{u}(\mathbf{r}) &\rightarrow 0 \end{aligned} \quad (5.120)$$

Note that incompressibility means $\partial V/\partial P = 0$, or that the density is independent of P . Therefore ∇P is arbitrary, and adjusts itself to whatever the boundary condition demands. Since there is no source for the fluid, we must have everywhere

$$\nabla \cdot \mathbf{u} = 0 \quad (5.121)$$

Taking the curl of both sides of (5.119), remembering that ρ is a constant, and neglecting terms of the form $(\partial \mathbf{u}/\partial x_i)(\partial \mathbf{u}/\partial x_j)$, we find that

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (\nabla \times \mathbf{u}) = 0 \quad (5.122)$$

i.e., that $\nabla \times \mathbf{u}$ is constant along a streamline. Note that P drops out because $\nabla \times (\nabla \times P) \equiv 0$. Since very far from the sphere we have $\nabla \times \mathbf{u} = 0$, it follows

that everywhere

$$\nabla \times \mathbf{u} = 0 \quad (5.123)$$

This means that \mathbf{u} is the gradient of some function:

$$\mathbf{u} = \nabla \Phi \quad (5.124)$$

where Φ is called the velocity potential. By (5.120) and (5.121) the equation and boundary conditions for Φ are

$$\begin{aligned} \nabla^2 \Phi(\mathbf{r}) &= 0 \\ \left(\frac{\partial \Phi}{\partial r} \right)_{r=a} &= u_0 \cos \theta \\ \Phi(\mathbf{r}) &\rightarrow 0 \quad r \rightarrow \infty \end{aligned} \quad (5.125)$$

where θ is the angle between u_0 and r , as shown in Fig. 5.6.

The most general solution to $\nabla^2 \Phi = 0$ is a superposition of solid harmonics.* Since the boundary condition involves $\cos \theta$, we try the solution

$$\Phi(\mathbf{r}) = A \frac{\cos \theta}{r^2} \quad (r \geq a) \quad (5.126)$$

which is a solid harmonic of order 1 and is the potential that would be set up if a dipole source were placed at the center of the sphere. Choosing $A = -\frac{1}{2}u_0a^3$ satisfies the boundary conditions. Therefore

$$\Phi(\mathbf{r}) = -\frac{1}{2}u_0a^3 \frac{\cos \theta}{r^2} \quad (r \geq a) \quad (5.127)$$

This is the only solution of (5.125), by the well-known uniqueness theorem of the Laplace equation. The velocity field of the fluid is then given by

$$\mathbf{u}(\mathbf{r}) = -\frac{1}{2}u_0a^3 \nabla \frac{\cos \theta}{r^2} \quad (r \geq a) \quad (5.128)$$

The streamlines can be sketched immediately, and they look like the electric field due to a dipole, as shown in Fig. 5.6.

Let us calculate the kinetic energy of the fluid. It is given by the integral

$$\begin{aligned} \text{K.E.} &= \int d^3r \frac{1}{2} \rho |\mathbf{u}|^2 = \frac{\rho}{2} \left(\frac{u_0 a^3}{2} \right)^2 \int_{r \geq a} d^3r \nabla \frac{\cos \theta}{r^2} \cdot \nabla \frac{\cos \theta}{r^2} \\ &= \frac{\rho}{2} \left(\frac{u_0 a^3}{2} \right)^2 \int_{r \geq a} d^3r \nabla \cdot \left[\frac{\cos \theta}{r^2} \nabla \frac{\cos \theta}{r^2} \right] \\ &= -\frac{\rho}{2} \left(\frac{u_0 a^3}{2} \right)^2 \int_{r=a} d\mathbf{S} \cdot \left[\frac{\cos \theta}{r^2} \nabla \frac{\cos \theta}{r^2} \right] \\ &= -\frac{\rho}{2} \left(\frac{u_0 a^3}{2} \right)^2 a^2 \int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos \theta) \left(\frac{\cos \theta}{r^2} \frac{\partial}{\partial r} \frac{\cos \theta}{r^2} \right)_{r=a} = \frac{1}{2} m' u_0^2 \end{aligned} \quad (5.129)$$

*A solid harmonic is $r^l Y_{lm}$ or $r^{-l-1} Y_{lm}$, where Y_{lm} is a spherical harmonic.

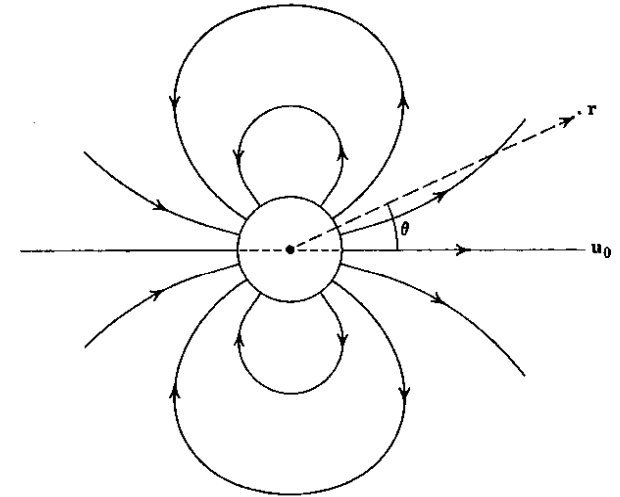


Fig. 5.6 Streamlines in a nonviscous liquid in the presence of a moving sphere.

where m' is half the mass of the displaced fluid:

$$m' = \frac{1}{2} \left(\frac{4}{3} \pi a^3 \rho \right) \quad (5.130)$$

If the sphere has a mechanical mass m , the total kinetic energy of the system of liquid plus sphere is

$$E = \frac{1}{2} (m_0 + m') u_0^2 \quad (5.131)$$

The mass $m_0 + m'$ may be interpreted to be the effective mass of the sphere, since (5.131) is the total energy that has to be supplied for the sphere to move with velocity u_0 .

Stokes' Law

We proceed to solve the same problem when the fluid has a nonvanishing coefficient of viscosity μ . The Navier-Stokes equation will be taken to be

$$0 = -\nabla \left(P - \frac{\mu}{3} \nabla \cdot \mathbf{u} \right) + \mu \nabla^2 \mathbf{u} \quad (5.132)$$

on the assumption that the material derivative of \mathbf{u} , which gives rise to the effective mass, is small compared to the viscous terms. We return to examine the validity of this approximation later. Since there is no source for the fluid, we still require $\nabla \cdot \mathbf{u} = 0$, and (5.132) becomes the simultaneous equations

$$\begin{aligned} \nabla^2 \mathbf{u} &= \frac{1}{\mu} \nabla P \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \quad (5.133)$$

with the boundary condition that the fluid sticks to the sphere. Let us translate the coordinate system so that the sphere is at rest at the origin while the fluid at infinity flows with uniform constant velocity \mathbf{u}_0 . The equations (5.133) remain invariant under the translation, whereas the boundary conditions become

$$\begin{aligned} [\mathbf{u}(\mathbf{r})]_{r=a} &= 0 \\ \mathbf{u}(\mathbf{r}) &\xrightarrow{r \rightarrow \infty} \mathbf{u}_0 \end{aligned} \quad (5.134)$$

Taking the divergence of both sides of the first equation of (5.133), we obtain

$$\nabla^2 P = 0 \quad (5.135)$$

Thus the pressure, whatever it is, must be a linear superposition of solid harmonics. A systematic way to proceed would be to write P as the most general superposition of solid harmonics and to determine the coefficient by requiring that (5.133) be satisfied. We take a short cut, however, and *guess* that P is, apart from an additive constant, a pure solid harmonic of order 1:

$$P = P_0 + \mu P_1 \frac{\cos \theta}{r^2} \quad (5.136)$$

where P_0 and P_1 are constants to be determined later. With this, the problem reduces to solving the inhomogeneous Laplace equation

$$\nabla^2 \mathbf{u} = P_1 \nabla \frac{\cos \theta}{r^2} \quad (5.137)$$

subject to the conditions

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ [\mathbf{u}(\mathbf{r})]_{r=a} &= 0 \\ \mathbf{u}(\mathbf{r}) &\xrightarrow{r \rightarrow \infty} \mathbf{u}_0 \end{aligned} \quad (5.138)$$

A particular solution of (5.137) is

$$\mathbf{u}_1 = -\frac{P_1}{6} r^2 \nabla \frac{\cos \theta}{r^2} = -\frac{P_1}{6} \left(\frac{\hat{z}}{r} - 3r \frac{z}{r^3} \right) \quad (5.139)$$

where \hat{z} denotes the unit vector along the z axis, which lies along \mathbf{u}_0 . It is easily verified that (5.139) solves (5.137), if we note that $1/r$ and z/r^3 are both solid harmonics. Thus,

$$\nabla^2 \mathbf{u}_1 = -\frac{P_1}{6} \left[-3\nabla^2 \left(\frac{rz}{r^3} \right) \right] = P_1 \nabla \left(\frac{z}{r^3} \right) = P_1 \nabla \frac{\cos \theta}{r^2} \quad (5.140)$$

The complete solution is obtained by adding an appropriate homogeneous solution to (5.139) to satisfy (5.138). By inspection we see that the complete solution is

$$\mathbf{u} = \mathbf{u}_0 \left(1 - \frac{a}{r} \right) + \frac{1}{4} u_0 a (r^2 - a^2) \nabla \frac{\cos \theta}{r^2} \quad (5.141)$$

where we have set

$$P_1 = -\frac{3}{2} u_0 a \quad (5.142)$$

to have $\nabla \cdot \mathbf{u} = 0$.

We now calculate the force acting on the sphere by the fluid. By definition the force per unit area acting on a surface whose normal point along the x_j axis is $-\mathbf{T}_j$ of (5.107). It follows that the force per unit area acting on a surface element of the sphere is

$$\mathbf{f} = -\left(\frac{x}{r} \mathbf{T}_1 + \frac{y}{r} \mathbf{T}_2 + \frac{z}{r} \mathbf{T}_3 \right) = -\hat{r} \cdot \vec{P} \quad (5.143)$$

where \hat{r} is the unit vector in the radial direction and \vec{P} is given by (5.118). The total force experienced by the sphere is

$$\mathbf{F}' = \int dS \mathbf{f} \quad (5.144)$$

where dS is a surface element of the sphere and the integral extends over the entire surface of the sphere. Thus it is sufficient to calculate \mathbf{f} for $r = a$.

The vector $\hat{r} \cdot \vec{P}$ has the components

$$\begin{aligned} (\hat{r} \cdot \vec{P})_i &= \frac{1}{r} x_j P_{ji} = \frac{1}{r} x_j \left[\delta_{ji} P - \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] \\ &= \frac{x_i}{r} P - \frac{\mu}{r} \left[\frac{\partial}{\partial x_i} (x_j u_j) - u_i + x_j \frac{\partial}{\partial x_j} u_i \right] \end{aligned}$$

Hence

$$\mathbf{f} = -\hat{r} P + \frac{\mu}{r} [\nabla(\mathbf{r} \cdot \mathbf{u}) - \mathbf{u} + (\mathbf{r} \cdot \nabla) \mathbf{u}] \quad (5.145)$$

where P is given by (5.136) and (5.142), and \mathbf{u} is given by (5.141). Since $\mathbf{u} = 0$ when $r = a$, we only need to consider the first and the last terms in the bracket. The first term is zero at $r = a$ by a straightforward calculation. At $r = a$ the second term is found to be

$$\frac{1}{r} [(\mathbf{r} \cdot \nabla) \mathbf{u}]_{r=a} = \left(\frac{\partial \mathbf{u}}{\partial r} \right)_{r=a} = \frac{3}{2} \frac{\mathbf{u}_0}{a} - \frac{3}{2} \frac{f u_0 \cos \theta}{a} \quad (5.146)$$

When this is substituted into (5.145), the second term exactly cancels the dipole part of $\hat{r} P$, and we obtain

$$(\mathbf{f})_{r=a} = -\hat{r} P_0 + \frac{3}{2} \frac{\mu}{a} \mathbf{u}_0$$

The constant P_0 is unknown, but it does not contribute to the force on the sphere. From (5.144) we obtain

$$\mathbf{F} = 6\pi\mu a \mathbf{u}_0 \quad (5.147)$$

which is Stokes' law.

The validity of (5.141) depends on the smallness of the material derivative of \mathbf{u} as compared to $\mu \nabla^2 \mathbf{u}$. Both these quantities can be computed from (5.141). It is then clear that we must require

$$\frac{\rho u_0 a}{\mu} \ll 1 \quad (5.148)$$

Thus Stokes' law holds only for small velocities and small radii of the sphere. A more elaborate treatment shows that a more accurate formula for \mathbf{F}' is

$$\mathbf{F}' = 6\pi\mu a \mathbf{u}_0 \left(1 + \frac{3}{8} \frac{\rho u_0 a}{\mu} + \dots \right) \quad (5.149)$$

The pure number $\rho u_0 a / \mu$ is called the Reynolds number. When the Reynolds number becomes large, turbulence sets in and streamline motion completely breaks down.

PROBLEMS

- 5.1** Make order-of-magnitude estimates for the mean free path and the collision time for
- H_2 molecules in a hydrogen gas in standard condition (diameter of $\text{H}_2 = 2.9 \text{ \AA}$);
 - protons in a plasma (gas of totally ionized H_2) at $T = 3 \times 10^5 \text{ K}$, $n = 10^{15}$ protons/cm³, $\sigma = \pi r^2$, where $r = e^2/kT$;
 - protons in a plasma at the same density as (b) but at $T = 10^7 \text{ K}$, where thermonuclear reactions occur;
 - protons in the sun's corona, which is a plasma at $T = 10^6 \text{ K}$, $n = 10^6$ protons/cc;
 - slow neutrons of energy 0.5 MeV in ^{238}U ($\sigma \approx \pi r^2$, $r \approx 10^{-13} \text{ cm}$).
- 5.2** A box made of perfectly reflecting walls is divided by a perfectly reflecting partition into compartments 1 and 2. Initially a gas at temperature T_1 was confined in compartment 1, and compartment 2 was empty. A small hole of dimension much less than the mean free path of the gas is opened in the partition for a short time to allow a small fraction of the gas to escape into compartment 2. The hole is then sealed off and the new gas in compartment 2 comes to equilibrium.
- During the time when the hole was open, what was the flux dI of molecules crossing into compartment 2 with speed between v and $v + dv$?
 - During the same time, what was the average energy per particle ϵ of the molecules crossing into compartment 2?
 - After final equilibrium has been established, what is the temperature T_2 in compartment 2?

Answer. $T_2 = \frac{4}{3} T_1$.

- 5.3** (a) Explain why it is meaningless to speak of a sound wave in a gas of strictly noninteracting molecules.
- (b) In view of (a), explain the meaning of a sound wave in an ideal gas.

5.4 Show that the velocity of sound in a real substance is to a good approximation given by $c = 1/\sqrt{\rho \kappa_S}$, where ρ is the mass density and κ_S the adiabatic compressibility, by the following steps.

(a) Show that in a sound wave the density oscillates adiabatically if

$$K \ll c \lambda \rho c_V$$

where K = coefficient of thermal conductivity

λ = wavelength of sound wave

ρ = mass density

c_V = specific heat

c = velocity of sound

(b) Show by numerical examples, that the criterion stated in (a) is well satisfied in most practical situations.

5.5 A flat disk of unit area is placed in a dilute gas at rest with initial temperature T . Face A of the disk is at temperature T , and face B is at temperature $T_1 > T$ (see sketch). Molecules striking face A reflect elastically. Molecules striking face B are absorbed by the disk, only to re-emerge from the same face with a Maxwellian distribution of temperature T_1 .

(a) Assume that the mean free path in the gas is much smaller than the dimension of the disk. Present an argument to show that after a few collision times the gas can be described by the hydrodynamic equations, with face B replaced by a boundary condition for the temperature.

(b) Write down the first-order hydrodynamic equations for (a), neglecting the flow of the gas. Show that there is no net force acting on the disk.

(c) Assume that the mean free path is much larger than the dimensions of the disk. Find the net force acting on the disk.

5.6 A square vane, of area 1 cm², painted white on one side, black on the other, is attached to a vertical axis and can rotate freely about it (see the sketch). Suppose the arrangement is placed in He gas at room temperature and sunlight is allowed to shine on the vane. Explain qualitatively why

(a) at high density of the gas the vane does not move;

(b) at extremely small densities the vane rotates;

(c) at some intermediate density the vane rotates in a sense opposite to that in (b). Estimate this intermediate density and the corresponding pressure.

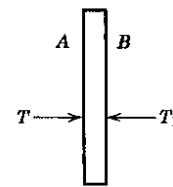


Fig. P5.5