Chapter 5

FRAGMENTATION

In fragmentation, an object continuously breaks into an increasing number of smaller pieces by external driving. Fragmentation is ubiquitous in nature. At geological scales, fragmentation is responsible for sand grains on beaches and for boulder fields. At the molecular level, chemical bond breaking underlies polymer degradation and the consumption of material in combustion. A fragmentation event can be visualized as

$$A_x \xrightarrow{a(x)} A_{x_1} + A_{x_2} + \ldots + A_{x_n},$$

in which an object of mass x breaks in a mass-conserving way at a rate $a(x)$ into n fragments of masses $x_1, x_2, \ldots, x_n$, with $B(x_i|x)$ the production rate of a daughter fragment of mass x_i from a fragment of mass x (Fig. 5.1). Here n may be fixed or varying in each breaking event. As in the complementary process of aggregation, we want to understand which microscopic features of the breakup rates $a(x)$ and $B(x_i|x)$ determine the distribution of fragment masses, $c(x,t)$.

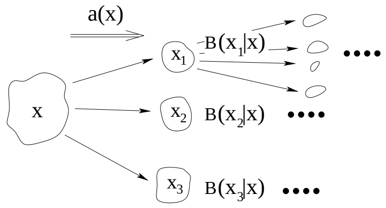

Figure 5.1: Schematic representation of fragmentation. A fragment of mass x breaks at an overall rate $a(x)$ into 3 daughter fragments of masses x_i, with respective production rates $B(x_i|x)$.

5.1 The Master Equation

The evolution of the mass distribution is described by the master equation

$$\frac{\partial c(x,t)}{\partial t} = -a(x)\, c(x,t) + \int_x^\infty c(y,t)\, a(y)\, B(x|y)\, dy\,. \tag{5.1}$$

The first term on the right accounts for the loss of fragments of mass x due to their breakup at overall rate $a(x)$. The second term accounts for the gain of fragments of mass x from the breakup of objects of mass larger than x. This master equation description involves a number of basic approximations, including:

- Homogeneity: fragment densities independent of spatial position. This is the mean-field assumption.
- Shape independence. Fragment shape plays no role in the evolution.
- Linearity. The breakup properties of a given cluster does not depend on the state of any other clusters.

Binary Breakup

Binary breakup refers to the specific case where two fragments are produced in each breaking event. The master equations may be written in a form that explicitly highlights this binary feature:

$$\frac{\partial c(x,t)}{\partial t} = -c(x,t)\int_0^x F(y,x-y)\,dy + 2\int_x^\infty c(y,t)\,F(x,y-x)\,dy\,. \tag{5.2}$$

Here $F(u,v)$ is the overall rate at which a fragment of mass $u+v$ breaks up into fragments of masses u and v. The factor 2 in the last term accounts for the fact that either one of the two daughter fragments from the breakup of a cluster of mass y may have mass x. Comparing with Eq. (5.1), we have $a(x) = \int_0^x F(y,x-y)\,dy$. Symmetry in the interchange of the two daughter fragments also implies that $B(x|y) = B(y-x|y)$. It is natural that the overall breakup rate does not depend on the masses of the two output fragments, in which case $F(y,x-y)$ is independent of y. Then the coefficient multiplying $c(x,t)$ in the first term on the right side of Eq. (5.2) — the overall breakup rate $a(x)$ — becomes $xf(x)$. Similarly, the coefficient of the second term is just $f(y)$. Perhaps the simplest example of binary breakup is *random scission*, where the breakup rate equals the fragment mass, $a(x) = x$ (equivalently, $f(x) = 1$). This choice describes the depolymerization of a linear polymer in which each chemical bond breaks at a fixed rate. Equivalently, random scission can be viewed as the cutting of a line (Fig. 5.2), with cuts occurring at a fixed rate per unit length. For random scission, the master equations reduce to

$$\frac{\partial c(x,t)}{\partial t} = -xc(x,t) + 2\int_x^\infty c(y,t)dy\,, \tag{5.3}$$

where we assume a scission rate of 1 per unit length.

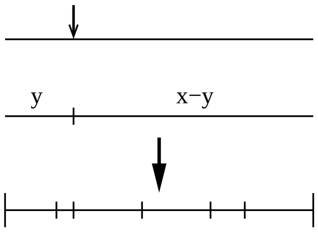

Figure 5.2: Random scission. An initial segment of length x is cut at a random point into two segments of lengths y and $x-y$. A series of such events can be viewed as the random deposition of cuts (bottom).

We now exploit this cutting picture to give a probabilistic derivation of the fragment mass distribution. Since cuts occur at random with rate 1, a segment of length Δx will be cut with probability $t\Delta x$ in a time t, and remains intact with probability $e^{-t\Delta x}$. There are then three types of fragments: (i) the initial segment remains intact, (ii) a fragment is at one end of the segment, (iii) a fragment is in the interior. We now evaluate the respective probabilities for these cases at time t, starting with an initial fragment of length L:

0 Intact segment. Since the average number of cuts is Lt, the segment remains intact with probability e^{-Lt}. The length distribution for such a segment is then $p_0(x) = e^{-Lt}\delta(x-L)$.

1 End fragment. An end fragment of length in the range $(x, x+dx)$ is created when there are no cuts within a distance x from the end (probability e^{-xt}), and the interval $(x, x+dx)$ contains a cut (probability $t\,dx$). The probability for this event is $p_1(x)dx = 2t\,e^{-xt}\,dx$, where the factor 2 accounts for the fragment being at either end. The length distribution for such segments is then $p_1(x) = 2te^{-xt}$.

2 Interior fragment. Such a fragment arises when there are two cuts in the intervals dx and dz at opposite ends of a otherwise intact fragment of length x. The probability for this event is $t^2\,dx\,dz\,e^{-xt}$. Integrating over all possible fragment positions $(x < z < L)$ gives $p_2(x) = t^2(L-x)e^{-xt}$.

Combining these three events gives the probability for a fragment of length x at time t:

$$c(x,t) = e^{-xt}\left\{\delta(L-x) + \left[2t + t^2(L-x)\right]\Theta(L-x)\right\}. \tag{5.4}$$

It is easy to check that this expression is a solution of the master equation (5.3).

Because of the linearity of the master equations, we may average Eq. (5.4) over an arbitrary initial distribution $c(x,t=0) \equiv c_0(L)$ to give the general mass distribution

$$c(x,t) = e^{-xt}\left\{c_0(x) + \int_x^\infty c_0(y)\left[2t + t^2(y-x)\right]dy\right\}. \tag{5.5}$$

The controlling factor in this solution is the exponential e^{-xt} which leads to the distribution being non-zero only for $x < t^{-1}$. Thus the typical fragment mass shrinks and concomitantly the number of segments grows with time. The role of the initial boundaries of the segment thus become irrelevant as interior segments predominate the mass distribution. Thus keeping only the contribution of interior segments and using the fact that the typical segment length approaches zero as $1/t$, the asymptotic mass distribution reduces to

$$c(x,t) \simeq L_0\, t^2\, e^{-xt}, \tag{5.6}$$

where $L_0 = \int x\, c_0(x)\, dx$ is the average initial fragment mass. Notice that this asymptotic form can be written in the scaling form $c(x,t) \simeq L_0 s^{-2} f(x/s)$, with typical mass $s(t) = t^{-1}$ and scaling function $f(z) = e^{-z}$. For an exponential initial condition, $c_0(x) = e^{-xs}$, Eq. (5.5) gives the fragment mass distribution $c(x,t) = (1+t/s)^2\, e^{-x(t+s)}$. Thus an exponential distribution is preserved under fragmentation, but with a width that continuously decreases with time.

An alternative approach for determining the mass distribution is the *Charlesby Method*, which is essentially a Taylor expansion of the distribution in time. Even though a Taylor expansion ostensibly applies for short times, a key feature is that the series expansion coefficients can be obtained recursively. Consequently this method is exact, as nicely illustrated by the random scission model. As suggested by the form of the master equation, it is convenient to write $c(x,t)$ as $e^{-xt}F(x,t)$, a construction that leads to a master equation for F that contains only a gain term. We therefore write the $c(x,t)$ as the power series

$$c(x,t) = e^{-xt}\sum_{k=0}^{\infty} f_k(x)\, t^k. \tag{5.7}$$

The lowest-order term is dictated by the initial condition, $f_0(x) = c_0(x)$. Substituting this expansion into (5.3) and differentiating once with respect to x, yields the recursion relation for the expansion functions

$$\frac{d}{dx} f_k(x) = \frac{k-3}{k} f_{k-1}(x). \tag{5.8}$$

Integrating the first two of these equations, $f_1' = -2f_0$ and $f_2' = -\frac{1}{2}f_1$, yields

$$f_1(x) = 2\int_x^\infty c_0(x)dx, \qquad f_2(x) = \int_x^\infty (y-x)c_0(y)dy. \tag{5.9}$$

The terms with $k \geq 3$ all vanish and the resulting solution coincides with (5.5).

Finally, we present a direct solution to the master equation. The form of the first two terms in Eq. (5.3) leads to the integrating factor $e^{x(t+B)}$, with B a constant. This observation suggests that we seek a solution of the form $c(x,t) = A(t)\, e^{-x(t+B)}$. At this stage, it is convenient to differentiate the master equation with respect to x to convert it to the partial differential equation $c_{xt} = -xc_x - 3c$, where the subscripts denote partial differentiation. Substituting the ansatz $c(x,t) = A(t)\, e^{-x(t+B)}$ into this equation gives $\dot{A} = 2A/(t+B)$, with solution $A(t) = (t+B)^2$. The fragment mass distribution thus has the form $c(x,t) \propto (t+B)^2\, e^{-(t+B)x}$ and by matching with the initial condition, we reproduce the solution given in Eq. (5.5).

By this direct approach, we can also treat homogeneous scission, in which the overall breakup rate of a cluster is $a(x) = x^\lambda$ for general λ. The master equation is now

$$\frac{\partial c(x,t)}{\partial t} = -x^\lambda\, c(x,t) + 2\int_x^\infty y^{\lambda-1} c(y,t)\, dy. \tag{5.10}$$

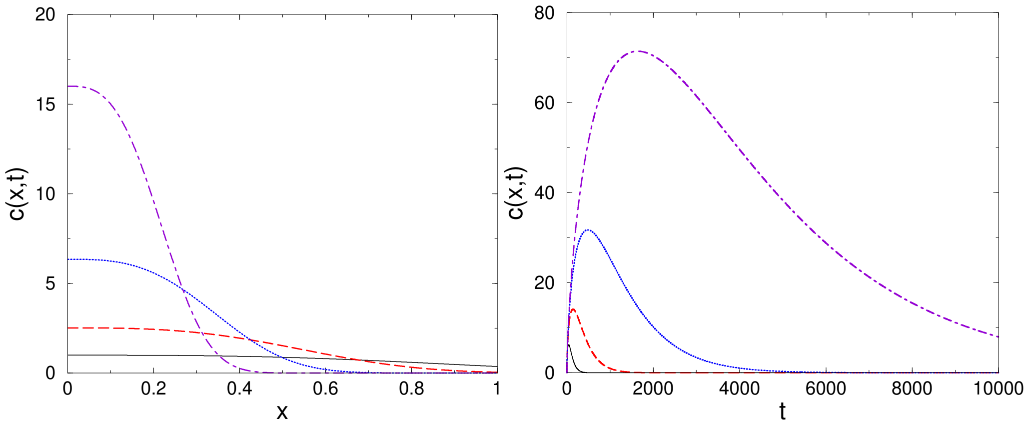

Figure 5.3: (Left) The asymptotic mass distribution (5.12) for the case $\lambda = 3$ as a function of x for $t = 1$ (solid), 4 (dash), 16 (dotted), and 64 (dot-dash). (Right) The same mass distribution as a function of t for $x = \frac{1}{4}$ (solid), $\frac{1}{6}$ (dash), $\frac{1}{9}$ (dotted), and $\frac{1}{13.5}$ (dot-dash).

We now seek a solution of the form $c(x,t) = A(t)e^{-x^\lambda(t+B)}$. Substituting this form into the master equation yields a soluble ordinary differential equation for $A(t)$, from which one obtains a special solution for $c(x,t)$,

$$c(x,t) = (1 + t/B_0)^{2/\lambda} e^{-(t+B_0)x^\lambda}. \tag{5.11}$$

By the linearity of the master equations, the general solution for $c(x,t)$ can be written as a linear combination, that is, an integral transform of Eq. (5.11), which, in turn, can be expressed in terms of special functions. In the long-time limit, the asymptotic behavior is

$$c(x,t) \propto t^{2/\lambda} \exp(-tx^\lambda). \tag{5.12}$$

A sketch of this mass distribution is shown in Fig. 5.3, both as a function of time for fixed mass, and as a function of mass at fixed times. Qualitatively, the behavior is in accord with simple-minded intuition. The population of a given (small) mass, x, initially grows due to the breakup of larger fragments. Eventually, however, the population at this mass decays when the production of x diminishes due to the depletion of larger fragments. This decay is asymptotically exponential, with the decay time varying inversely in the particle mass. For the distribution at fixed time, there is a steepening near the origin as a function of time, reflecting the eventual predominance by very small fragments. This small mass tail often has a power law form, as discussed previously. Notice also that this asymptotic form for $c(x,t)$ becomes pathological as $\lambda \to 0$. This is a signal of the shattering transition, which will be discussed in a later section.

A particularly interesting special case is when the homogeneity exponent equals zero. For $F(x,y) \propto (x+y)^{-1}$, the exact solution for the fragment mass distribution in the limit $\lambda \to 0$ is,

$$c(x,t) = e^{-t}\delta(x-l) + \frac{2te^{-t}}{l} \sum_{n=0}^{\infty} \frac{[2t\ln(l/x)]^n}{n!\,(n+1)!}. \tag{5.13}$$

In the small mass limit, a singularity in $c(x,t)$ develops due to the explosive growth in the number of very small fragments. The corresponding moments of this mass distribution are,

$$M_n(t) = l^n \exp[(1-n)t/(1+n)]. \tag{5.14}$$

Thus the total number of fragments, $M_0(t)$, grows exponentially in time, in contrast to the power law growth for $M_0(t)$ in the non-shattering regime. As we shall see, this solution corresponds to a system on the borderline between scaling and shattering.

5.2 Scaling

We now present the scaling approach to determine the asymptotic behavior of the fragment mass distribution. As in aggregation, the applicability of scaling rests on the hypothesis that a well-defined continuously-decreasing typical mass scale exists. Consequently, the mass distribution should be a function *only* of the ratio of the mass of a fragment to this typical mass. As in aggregation, we assume that the breakup rates are homogeneous functions: the overall breakup rate is $a(x) = x^\lambda$, thereby defining the homogeneity index λ, and the daughter rate $B(x|y)$ depends only on the ratio of the daughter mass to initial fragment mass, so that $B(x|y) \propto y^{-1}\, b(\frac{x}{y})$. By definition, the integral $\int_0^1 b(x)\, dx$ equals the average number of fragments produced in a single breakup event, and mass conservation imposes the condition $y = \int_0^y x\, B(x|y)\, dx$. In scaled form, this latter statement becomes $\int_0^1 x\, b(x)\, dx = 1$.

Let's start by investigating the moments of the mass distribution. We define the $\alpha^{\rm th}$ moment of this distribution, $M_\alpha(t) \equiv \int_0^\infty x^\alpha\, c(x,t)\, dx$, where the order α will typically be non-integer. To determine the time dependence of M_α, we multiply the master equation (5.1) by x^α and integrate over all x to give

$$\int_0^\infty \left[x^\alpha \frac{\partial c(x,t)}{\partial t} = -x^{\alpha+\lambda} c(x,t) + x^\alpha \int_x^\infty c(y,t) y^{\lambda-1} b(x/y)\, dy \right]. \tag{5.15}$$

We now apply a simple trick, that we will use repeatedly, to yield a closed equation for the moments. We simply interchange the integration order in the last term so that each integral involves a single variable (see Fig. 5.4). This interchange gives

$$\begin{aligned}
\int_0^\infty x^\alpha\, dx \int_x^\infty c(y) y^{\lambda-1} b(x/y)\, dy &= \int_0^\infty y^{\lambda-1} c(y)\, dy \int_0^y x^\alpha b(x/y) dx \\
&= \int_0^\infty y^{\alpha+\lambda} c(y)\, dy \int_0^1 z^\alpha b(z)\, dz \qquad z \equiv x/y \\
&= M_{\alpha+\lambda} L_\alpha,
\end{aligned}$$

where $L_\alpha \equiv \int_0^1 x^\alpha b(x)\, dx$ is the $\alpha^{\rm th}$ moment of the scaled daughter breakup rate.

Figure 5.4: Illustration of the interchange of integration order. Left: Integrating first over y from x to ∞ and then over all x. Right: Integrating first over x from 0 to y and then over all y.

The moments therefore evolve as

$$\dot{M}_\alpha = (L_\alpha - 1)\, M_{\alpha+\lambda}. \tag{5.16}$$

Starting with $M_1(t) = 1$, Eq. (5.16) gives $\dot{M}_{1-\lambda}(t) = (L_{1-\lambda} - 1)$, with solution $M_{1-\lambda}(t) = (L_{1-\lambda} - 1)t + c$, where $L_{1-\lambda} - 1 > 0$ for $\lambda < 1$ and c is another constant of order 1. Iterating leads to the asymptotic solution,

$$M_{1-k\lambda} \simeq \prod_{j=1}^{k} (L_{1-j\lambda} - 1)\, \frac{t^k}{k!} \;\propto\; t^k, \tag{5.17}$$

for the discrete set of index values $1 - k\lambda$. Thus the negative moments, $M_{1-k\lambda}$, with k a positive integer, are connected recursively and asymptotically grow as t^k. These negative moments play the role analogous to the positive integer moments of the mass distribution in aggregation.

We now apply scaling to determine the asymptotics of the fragment mass distribution itself. As in aggregation [see Eq. (4.64)], we write the scaling ansatz for the mass distribution in fragmentation as

$$c(x,t) = \frac{1}{s^2}\, f\!\left(\frac{x}{s}\right), \tag{5.18}$$

where $s(t)$ is the typical fragment mass, and the exponent -2 enforces mass conservation. As already highlighted in the corresponding discussion of the scaling theory of aggregation (Sec. 4.2), the basic feature of scaling for the mass distribution is it is a function of the ratio of the mass to the typical mass, rather than depending on mass and time separately. We now substitute the scaling ansatz (5.18) into the master equation (5.1) to give, after some simple algebra,

$$\frac{\dot{s}}{s^3}[f(u)+uf'(u)] = s^{\lambda-2}\left[-u^{\lambda}\, f(u) + \int_u^{\infty} f(v)\, v^{\lambda-1}\, b\Big(\frac{u}{v}\Big)\, dv\right].$$

Now it is a simple matter to separate out the time and scaled mass dependences to give

$$\begin{aligned} \dot{s}\, s^{-(1+\lambda)} &= -\omega \\ \omega[2f(u)+u\, f'(u)] &= -u^{\lambda}\, f(u) + \int_u^{\infty} f(v)\, v^{\lambda-1}\, b\Big(\frac{u}{v}\Big)\, dv, \end{aligned} \tag{5.19}$$

where ω is the separation constant that is positive so that the typical mass decreases with time.

From the first equation, the typical fragment mass has the asymptotic time dependence

$$s(t) \sim \begin{cases} (\lambda\omega t)^{-1/\lambda}, & \text{for } \lambda > 0, \\ e^{-\omega t}, & \text{for } \lambda = 0, \\ (t_c - t)^{1/|\lambda|}, & \text{for } \lambda < 0 \text{ and } t \to t_c, \end{cases} \tag{5.20}$$

where $t_c = [s(0)\lambda\omega]^{-1}$. Here is a nice demonstration of the utility of the scaling approach: with minimal knowledge about system details, we've determined the time dependence of the average fragment mass with little labor! The last case is particularly striking: when λ is less than zero, the fragmentation of the tiniest fragments occurs so quickly that the mean fragment mass vanishes at a finite time t_c. This singularity signals the *shattering* transition and is analogous to the gelation transition in aggregation with a homogeneity index larger than one. For $t > t_c$, a finite fraction of the mass is converted into a "dust" phase that consists of an infinite number of zero mass particles that manages to contain a finite fraction of the total mass.

Let's now determine the scaled fragment mass distribution. We start by converting the scaling equation for $f(u)$ [the second of Eqs. (5.19)] in the regime $\lambda > 0$ to a recursion for an infinite set of moments of the distribution, from which we then reconstruct the mass distribution. Thus we multiply both sides of Eq. (5.19) by u^{α} and integrate over all u. After an integration by parts, the left-hand side is simply $\omega(1-\alpha)m_{\alpha}$, where

$$m_{\alpha} \equiv \int_0^{\infty} x^{\alpha}\, f(x)\, dx \tag{5.21}$$

is the α^{th} moment of the scaling function. For the right-hand side, we interchange the integration order to transform (see again Fig. 5.4)

$$\int_0^{\infty} u^{\alpha}\, du \int_u^{\infty} v^{\lambda-1}\, f(v)\, b(u/v)\, dv \qquad \text{to} \qquad \int_0^{\infty} v^{\lambda-1}\, f(v)\, dv \int_0^{v} u^{\alpha}\, b(u/v)\, du, \tag{5.22}$$

and then express the second integral in terms of the moments of the scaled daughter breakup rate $L_{\alpha} = \int_0^1 x^{\alpha} b(x)\, dx$. We thereby obtain the moments of the scaling function recursively, in analogy with Eq. (5.16),

$$m_{\alpha+\lambda} = \omega\, \frac{\alpha-1}{1-L_{\alpha}}\, m_{\alpha}. \tag{5.23}$$

Comparing the definitions of the "bare" moments M_{α} and the scaled moments m_{α}, we see that they are related by $M_{\alpha}(t) = m_{\alpha} \times s(t)^{\alpha-1}$. The special case $\alpha = 0$ yields $N = M_0 = m_0/s$; that is, the average number of fragments N is inversely proportional to the typical mass.

To probe the large-mass tail of the mass distribution, one should focus on the high-order moments, that is, m_{α} for $\alpha \gg 1$. The most useful quantities are not the positive integer moments, but rather, from Eq. (5.23), the moments of order $\alpha = n\lambda$, with n a positive integer. For these α values, we iterate Eq. (5.23) and use the initial condition $m_0 = 1$ as well as the constraint $L_0 = 2$ for binary fragmentation, to give

$$m_{\alpha} = \omega^n \prod_{k=1}^{n-1} \frac{k\lambda - 1}{1 - L_{k\lambda}}. \tag{5.24}$$

For large n, the product is dominated by the factors with large k. In this limit, the form of $L_{k\lambda}$ is determined by the behavior of the breakup kernel $b(x)$ for $x \to 1$, namely, the limit where a fragment remains nearly intact in a breaking event. For this limit, a generic form of the kernel is $b(x) = b + \mathcal{O}((1-x)^\mu)$, for $x \to 1$, where $b \geq 0$ is the probability for no breaking to occur, and $\mu > 0$. Then $L_\alpha \simeq b/\alpha$ for large α. We now use the results of the highlight below to give

$$m_\alpha = \omega^n \lambda^{n-1} \frac{\Gamma(n-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})} \frac{\Gamma(n)\Gamma(1-\frac{b}{\lambda})}{\Gamma(n-\frac{b}{\lambda})}\,.$$

We now apply Eq. (4.3) to the ratio of gamma functions, as well as Stirling's approximation for $\alpha = n\lambda \to \infty$, to find after several straightforward steps,

$$m_\alpha \propto \left(\frac{\omega\alpha}{e}\right)^{\alpha/\lambda} \alpha^{[(b-1)/\lambda]-1/2}\,, \tag{5.25}$$

where a messy overall factor of order 1 has not been written.

Finite Products and Gamma Functions

We often encounter products of the form in Eq. (5.24) and are interested in this product for large n. What is the most direct way to obtain this asymptotics? One good way is to write the product in terms of gamma functions. For example, the numerator in (5.24) is

$$\prod_{k=1}^{n-1}(k\lambda-1) = \lambda^{n-1}\prod_{k=1}^{n-1}(k-\tfrac{1}{\lambda}) = \lambda^{n-1}\frac{\Gamma(n-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})}\,.$$

An important step is to factor out a constant such that each successive term in the product differs by 1, which can then be expressed as a ratio of gamma functions. Similarly for the denominator in (5.24), we use $L_{\kappa\lambda} \sim b/k\lambda$ to give

$$\prod_{k=1}^{n-1}(1-L_{k\lambda}) = \prod_{k=1}^{n-1}(1-\tfrac{b}{k\lambda}) = \frac{1}{\Gamma(n)}\prod_{k=1}^{n-1}(k-\tfrac{b}{\lambda}) = \frac{\Gamma(n-\frac{b}{\lambda})}{\Gamma(n)\Gamma(1-\frac{b}{\lambda})}\,.$$

To extract the mass distribution from these moments, we work backwards and determine what distribution leads to the moments gives above. Because the moments involve the gamma functions, it is natural to anticipate that the scaled mass distribution has the form $f(x) = Ax^\mu \exp(-ax^\nu)$, where a, μ, and ν are constants, and normalization gives $A = \nu\, a^{(1+\mu)/\nu}/\Gamma(\frac{\mu+1}{\nu})$. For this distribution, the moments are then

$$m_\alpha = a^{-\alpha/\nu}\,\frac{\Gamma(\frac{\alpha+\mu+1}{\nu})}{\Gamma(\frac{\mu+1}{\lambda})} \longrightarrow \left(\frac{\omega\alpha}{e}\right)^{\alpha/\nu} \alpha^{[(\mu+1)/\nu]-1/2} \qquad \alpha\to\infty, \tag{5.26}$$

when we choose $a = 1/(\lambda\omega)$, $\mu = b-2$, and $\nu = \lambda$. The latter form then matches Eq. (5.25). Thus we infer that the scaled mass distribution has the asymptotic behavior

$$f(x) \sim x^{b-2}\exp[-x^\lambda/(\lambda\omega)] \qquad x\to\infty, \tag{5.27}$$

from which the fragment mass distribution is

$$c(x,t) \propto \frac{x^{b-2}}{s^b}\, e^{-t\,x^\lambda}\,. \tag{5.28}$$

This asymptotics matches the exact behavior from homogeneous binary fragmentation.

To determine the small-mass tail of the mass distribution, we now need the moments with large negative order. Accordingly, we choose $\alpha = 1-n\lambda$ in Eq. (5.23) and iterate to arrive at the counterpart of Eq. (5.24),

$$m_{1-n\lambda} = \omega^{-n}\prod_{k=1}^{n}\frac{(L_{1-k\lambda}-1)}{k\lambda}\,. \tag{5.29}$$

Now the n dependence of $m_{1-n\lambda}$ for large n is now determined by the limiting form of $b(x)$ for x near 0, and there are two generic cases: breakup kernels with a strict lower cutoff and those without. A simple way to realize the former case, is to impose the condition that no clusters below a fixed relative mass $x_0 < 1$ are produced in a single breakup event. From the definition of L_α, the moment $L_{1-\alpha}$ then has the leading behavior $x_0^{-\alpha}/\alpha^{1+\mu}$ for large α. Substituting this expression into Eq. (5.29) yields

$$m_{1-n\lambda} \sim \frac{1}{(n!)^2\,(\omega\lambda^2)^n}\; x_0^{-\lambda-2\lambda-\cdots-n\lambda} \propto x_0^{-n^2\lambda/2} \sim e^{-\alpha^2\ln x_0/2\lambda} \qquad \alpha\to\infty.$$

We again work backwards and determine the form of the mass distribution that corresponds to these moments. The quadratic dependence on α in the exponential suggests that the distribution is log-normal, and it is immediate to verify that indeed the asymptotic behavior of $f(x)$ is given by

$$f(x) \sim e^{-\lambda(\ln x)^2/(2\ln x_0)} \qquad x\to 0. \tag{5.30}$$

This log-normal form is to be expected and it also follows from a classic argument that makes use of a correspondence between fragmentation and a random multiplicative process. Schematically the mass of a particular fragment evolves as $x_0 \to x_1 \to x_2 \to \cdots \to x_N$, where the successive reduction factor, $r_k = x_k/x_{k-1}$, is a random variable with a well-behaved distribution. By the central limit theorem, $\log x_N = \sum_{k=0}^N \log r_k$ will be normally distributed, so that x_N will be distributed log-normally.

A second general class of mass distributions arises for breakup rates in which arbitrarily small clusters can be produced in a single event. This behavior is typified by the power law daughter breakup rate $b(x)\sim x^\nu$ for small x. From Eq. (5.23), m_α diverges whenever L_α diverges, which occurs for $\alpha < \alpha_c < 0$, since m_0 is finite. For α close to α_c the leading term in Eq. (5.23) dominates to give $m_\alpha \propto L_\alpha$. By comparing the definitions $m_\alpha = \int x^\alpha f(x)\,dx$ and $L_\alpha = \int x^\alpha b(x)\,dx$, it follows that $f(x)$ coincides with $b(x)$. Therefore if the daughter breakup rate has a power law behavior for $x\to 0$, then $f(x)$ has the *same* power-law tail.

5.3 Fragmentation with Input

Fragmentation with steady material input arises in many industrial processes, such as the crushing of a steady stream of raw mineral ore. As in the complementary problem of aggregation with input, we want to understand how input influences the fragment mass distribution. Let the input rate of objects of mass x be $I(x)$; for convenience we set the total input rate to one, $\int x\,I(x)\,dx = 1$. We analyze the effect of input for the simple case of random scission with overall breakup rate $a(x) = x$. The master equation is now

$$\frac{\partial c(x,t)}{\partial t} = -xc(x,t) + 2\int_x^\infty c(y,t)dy + I(x). \tag{5.31}$$

It is again convenient to analyze this master equation by studying the evolution of the moments of the mass distribution. Here we we study not the moments *per se*, but rather, the *Mellin transform* of the mass distribution. The Mellin transform is the same as the moment except for a shift in its order (see the highlight about the Mellin transform on the next page). The Mellin transform is often more convenient than the moments and we will use this transform in the rest of this chapter. Now using the same interchange of integration order illustrated in Fig. 5.4, the Mellin transform $c(s,t) = \int c(x,t)\,x^{s-1}dx$ evolves according to

$$\frac{\partial c(s,t)}{\partial t} = \left(\frac{2-s}{s}\right) c(s+1,t) + I(s), \tag{5.32}$$

where $I(s) = \int I(x)x^{s-1}dx$, is the Mellin transform of the input rate. The Mellin transform is well suited to deal with integral equations of the form (5.31) that typifies fragmentation because it converts the integral equation into a recursion formula that is often soluble. A similar philosophy was the basis for the generating function approach in aggregation, where a convolution is converted to a product by the generating function.

In the steady state, the Mellin transform in (5.31) satisfies (after shifting the index s by 1)

$$c(s) = \left(\frac{s-1}{s-3}\right) I(s-1). \tag{5.33}$$

Using basic properties of the Mellin transform given in the highlight below, the steady-state fragment mass distribution is

$$c(x) = x^{-1} I(x) + 2x^{-3} \int_x^\infty I(y) y \, dy. \tag{5.34}$$

In the limit of small fragment masses, the second term becomes dominant. Since the integral approaches 1 as $x \to 0$, the mass distribution has the universal algebraic tail

$$c(x) \simeq 2x^{-3} \,, \tag{5.35}$$

that is *independent* of the details of the input for well-behaved input functions.

This asymptotic behavior can also be obtained by the following heuristic argument. Because of the steady input, the total mass in the system, $M(t) = c(s = 2, t)$, grows linearly with time, $M(t) = t$. Similarly, from Eq. (5.31), the total number of fragments $N(t) = c(s = 1, t)$ satisfies $\dot{N}(t) = t + \mu$, where $\mu = \int I(x)dx$ is the number of fragments added per unit time. Consequently, $N(t) = \frac{1}{2}t^2 + \mu t$. The first two moments imply that the typical fragment mass is $M/N \sim t^{-1}$. Thus the mass distribution should approach the scaling form

$$c(x,t) \simeq t^3 F(xt) \qquad \text{as } t \to \infty, \tag{5.36}$$

with prefactor t^{-3} to ensure that the total mass in the system grows linearly with time. With this scaling form, a steady state is possible only when $F(z) \sim z^{-3}$ for small x. This fact implies that $c(x) \sim x^{-3}$.

The Mellin Transform

The Mellin transform $M(s)$ of a function $c(x)$ and its inverse are defined by

$$M(s) = \int_0^\infty c(x)\, x^{s-1}\, dx, \qquad c(x) = \int_{c-i\infty}^{c+i\infty} x^{-s} M(s)\, ds. \tag{5.37}$$

The Mellin transform is just the moment of order $s-1$ of the function $c(x)$ and appears when one is interested in the moments of the probability distribution for a positive definite quantity. The Mellin transform is also just a Laplace transform in disguise. By defining $x = e^{-y}$, Eq. (5.37) immediately becomes the Laplace transform $M(s) = \int_0^\infty c(y)\, e^{-sy}\, dy$. Thus we can adapt the known rules about inverting the Laplace transform to infer the inverse of Mellin transforms.

Here are some basic rules about the Mellin transform and its inverse:

1. If the moments m_α are finite for $\alpha < \alpha_c$ and infinite for $\alpha > \alpha_c$, then $c(x)$ asymptotically behaves as the power law, $c(x) \propto x^{-1-\alpha_c}$.
2. If $M(s)$ is the Mellin transform of $c(x)$, then $M(s-n)$ is the Mellin transform of $x^{-n}c(x)$. It is easily verified that this relation is obvious.
3. The Mellin transform of $g(x) = x^{-m} \int_x c(y) y^{m-1} dy$ is $\frac{M(s)}{s-m}$. This result relies on the interchange of integration order illustrated in Fig. 5.4. With this interchange, the Mellin transform of $g(x)$ is

$$\begin{aligned}
\int_0^\infty x^{s-1}\, g(x)\, dx &= \int_0^\infty x^{s-1}\, x^{-m}\, dx \int_x^\infty y^{m-1}\, f(y)\, dy \\
&= \int_0^\infty y^{m-1}\, f(y)\, dy \int_0^\infty x^{s-m-1}\, dx \\
&= \frac{1}{s-m} \int_0^\infty y^{s-1}\, f(y)\, dy = \frac{1}{s-m} M(s)
\end{aligned}$$

The full time-dependent solution can be obtained by the Charlesby method. We start by expanding the Mellin transform as a power series in time

$$c(s,t) = \sum_{k=1}^\infty \frac{t^k}{k!}\, c_k(s)\,, \tag{5.38}$$

and then solve the expansion functions $c_k(s)$ iteratively. For an initially empty system, this expansion does not contain a constant term, while for a steady input, the Mellin transform of the input contains only a time-independent term $I(s)$. We now substitute the expansion (5.38) into Eq. (5.32) and equate terms with the same powers of time. This yields $c_1(s) = I(s)$, and $c_{k+1}(s) = -\frac{s-2}{s} c_k(s+1)$ for $k \geq 2$. Solving this set of equations recursively gives

$$c_{k+1}(s) = (-1)^k \frac{(s-1)(s-2)}{(s+k-1)(s+k-2)} I(s+k).$$

To invert this Mellin transform, we re-write $c_{k+1}(s)$ as the partial fraction expansion

$$c_{k+1}(s) = (-1)^k \left[1 - \frac{k(k+1)}{s+k-1} + \frac{k(k-1)}{s+k-2}\right] I(s+k).$$

From Eq. (5.38), the mass distribution can be written as a power series

$$c(x,t) = \sum_{k=0}^{\infty} \frac{t^{k+1}(-x)^k}{(k+1)!} c_k(x), \tag{5.39}$$

where the inverse transform of $c_{k+1}(s)$ has been conveniently written as $(-x)^k c_k(x)$. The three terms in the above expression for $c_{k+1}(s)$ can be inverted using the rules outlined in the Mellin transform highlight on page 83. The final expression for $c_k(x)$ is

$$c_k(x) = I_1(x) + \frac{k(k+1)}{x} I_2(x) + \frac{k(k-1)}{x^2} I_3(x), \tag{5.40}$$

with

$$I_1(x) = I(x) \quad I_2(x) = \int_x^\infty I(y)\,dy \quad I_3(x) = \int_x^\infty y f(y)\,dy. \tag{5.41}$$

Summing the three terms separately gives the mass distribution

$$c(x,t) = \sum_{k=1}^{3} t^k I_k(x) F_k(xt), \tag{5.42}$$

with the scaling functions

$$\begin{aligned} F_1(z) &= z^{-1}(1 - e^{-z}), \\ F_2(z) &= e^{-z}, \\ F_3(z) &= z^{-3}\left[2 - (2 + 2z + z^2)e^{-z}\right]. \end{aligned} \tag{5.43}$$

The function $F_3(z)$ has been obtained from the power series $F_3(z) = \sum_{k\geq 0} \frac{(-z)^k}{k!(k+3)}$. Eqs. (5.42) and (5.43) give the full time-dependent solution for an *arbitrary* time-independent input $I(x)$. In the limit $x \to 0$ and $t \to \infty$, with the scaling variable $z = xt$ kept finite, the third term in the sum of Eq. (5.42) dominates, and the anticipated scaling behavior of (5.36) is recovered with $F(z) = F_3(z)$.

5.4 Rectangular Fragmentation

It is natural to think of fragmentation as a geometric process, especially when one drops a plate that shatters when it hits the floor. Here we discuss a fragmentation model that incorporates the effects of shape in a minimalist way — rectangular fragmentation. A striking feature of this rectangular fragmentation is that satisfies an infinite number of "hidden" conservation laws. As a consequence, basic features of the mass distribution *multifractal* scaling (see the highlight on page 87 for a brief introduction).

Rectangular fragmentation is defined by the following breaking rule: given a population of rectangles, one is chosen with probability proportional to its area. For the selected rectangle, a random point inside the rectangle is chosen and then a crack through this point — either horizontal or vertical with equal probability

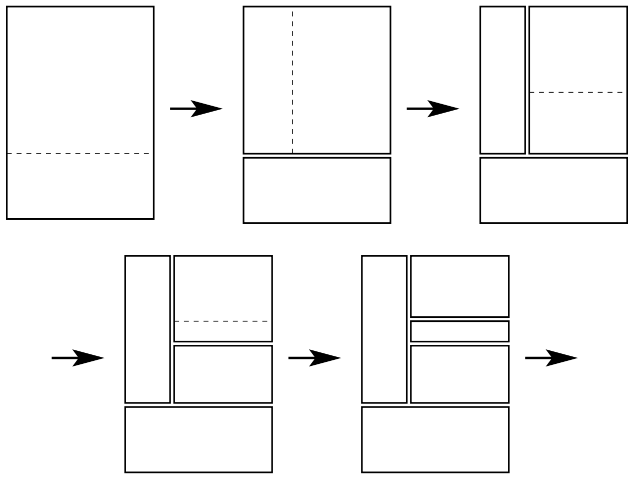

Figure 5.5: First few steps in rectangular fragmentation in two dimensions.

— fragments the rectangle into two smaller rectangles in an area-conserving manner (Fig. 5.5). Since one fragment is destroyed but two are created in each breaking event, the average number of fragments simply equals $N = 1 + t$.

The population of rectangles may be characterized by $c(x_1, x_2, t)$, the probability for a rectangle of length x_1 and width x_2 at time t. This distribution evolves in time according to the master equation

$$\frac{\partial c(x_1, x_2, t)}{\partial t} = -x_1 x_2\, c(x_1, x_2, t) + x_2 \int_{x_1}^{\infty} c(y_1, x_2, t)\, dy_1 + x_1 \int_{x_2}^{\infty} c(x_1, y_2, t)\, dy_2. \tag{5.44}$$

The first term accounts for the loss of a rectangle of area $A = x_1 x_2$ due to its fragmentation. Since the breaking point is chosen randomly inside from among the total area, the overall breaking rate is proportional to this area. The second term accounts for the gain of a fragment of length x_1 and height x_2 by vertically cracking a fragment of length $y_1 > x_1$ and height x_2. The prefactor x_2 accounts for the fact that the breaking point can be situated anywhere along the vertical crack. The last term accounts for the contribution due to a horizontal crack.

To solve (5.44), we introduce the 2-variable Mellin transform $M(s_1, s_2, t) \equiv \int\int x_1^{s_1-1} x_2^{s_2-1} c(x_1, x_2)\, dx_1\, dx_2$. Multiplying both sides of the master equation by $x_1^{s_1-1} x_2^{s_2-1}$ and integrating over all x_1 and x_2, the Mellin transform obeys the linear, non-local equation

$$\frac{\partial M(s_1, s_2, t)}{\partial t} = \left(\frac{1}{s_1} + \frac{1}{s_2} - 1\right) M(s_1 + 1, s_2 + 1, t). \tag{5.45}$$

From the definition of the Mellin transform, the moments of the fragment masses are then given by

$$\langle x_1^{n_1} x_2^{n_2} \rangle \equiv \frac{M(n_1 + 1, n_2 + 1)}{M(1, 1)}. \tag{5.46}$$

Thus, for example, the total area A of all fragments is just $M(2, 2)$, which is manifestly conserved in Eq. (5.45), while the average area is $\langle A \rangle = M(2, 2)/M(1, 1)$. Similarly, the total number of fragments $N = M(1, 1)$, satisfies $\frac{dN}{dt} = 1$, so that $N = (1 + t)$. Area conservation then implies that the average fragment area is $\langle A \rangle = \langle x_1 x_2 \rangle = (1 + t)^{-1}$.

A striking consequence of Eq. (5.45) is the existence of an infinity of conservation laws that are determined by the condition $(s_1^*)^{-1} + (s_2^*)^{-1} = 1$. These hidden conservation laws hold in an average sense. That is, while $M(s_1^*, s_2^*)$ does not strictly remain constant in each individual breaking event, $M(s_1^*, s_2^*)$, averaged over all realizations of rectangular fragmentation is conserved. However, the total area, $M(2, 2)$, is strictly conserved event by event.

Since the average mass decays algebraically with time in single-variable fragmentation with homogeneous breakup kernels, it is natural to assume that the Mellin transform also decays algebraically in time for rectangular fragmentation; that is $M(s_1, s_2) \sim t^{-\alpha(s_1,s_2)}$. Substituting this form into (5.45) gives $\alpha(s_1 + 1, s_2+1) = \alpha(s_1, s_2)+1$. The exponent $\alpha(s_1, s_2)$ can now be determined by combining this recursion for the exponent with the conservation law $\alpha(s_1^*, s_2^*) = 0$ to give

$$\alpha(s_1^* + k, s_2^* + k) = k \qquad \text{for all} \qquad (s_1^*)^{-1} + (s_2^*)^{-1} = 1. \tag{5.47}$$

Thus the value of α at an arbitrary point (s_1, s_2) is just the horizontal (or vertical) distance from this point to the curve $(s_1)^{-1} + (s_2)^{-1} = 1$ (Fig. 5.6). This distance condition gives $(s_1 - \alpha) = s_1^*$ and $(s_2 - \alpha) = s_2^* = [1 - (s_1^*)^{-1}]^{-1}$. Eliminating s_1^* from these two equations, the exponent is the smaller root of the quadratic equation $(s_1 - \alpha)(s_2 - \alpha) - (s_1 - \alpha) - (s_2 - \alpha) = 0$, which gives

$$\alpha(s_1, s_2) = \frac{s_1 + s_2}{2} - 1 - \sqrt{\frac{(s_1 - s_2)^2}{4} + 1}\ .$$

Consequently, the asymptotic behavior of the moments is

$$\langle x_1^{s_1} x_2^{s_2} \rangle \equiv \frac{M(s_1 + 1, s_2 + 1)}{M(1, 1)} \sim t^{-\alpha(s_1+1,s_2+1)+\alpha(1,1)} \sim t^{-\alpha(s_1,s_2)-2}\ . \tag{5.48}$$

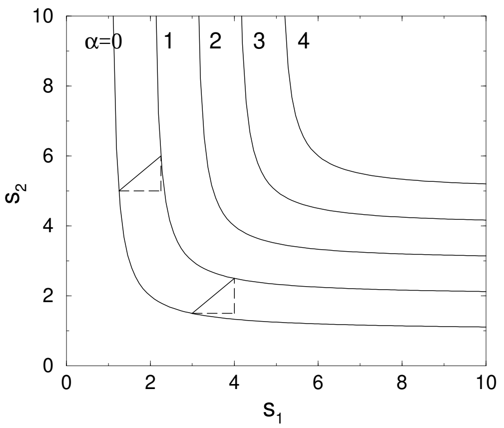

Figure 5.6: Loci of $\alpha(s_1, s_2) = 0$, 1, 2, 3, and 4 from Eq. (5.47). Each successive curve is shifted horizontally and vertically by 1 from its predecessor (dashed lines).

We now ask: what combinations of fragment length and width are conserved in rectangular fragmentation? The requirement that $M(s_1, s_2)$ is constant implies that $(s_1)^{-1} + (s_2)^{-1} = 1$. Therefore moments of the form

$$\int_0^\infty \int_0^\infty x_1^k\, x_2^{1/k}\, c(x_1, x_2)\, dx_1\, dx_2$$

are conserved for k. The case $k = 1$ gives the obvious conservation of the total area, but other k values lead to simple moment combinations whose physical meaning is not yet understood.

The non-linear spectrum of scaling exponents $\alpha(s_1, s_2)$ is an example of *multiscaling* or *multifractal scaling* in which high-order moments are not simply related to low-order moments, *viz.* $\langle x^n \rangle \gg \langle x \rangle^n$. A nice illustration of the consequences of this multiscaling arises in the moments of the fragment length:

$$\langle \ell^n \rangle \equiv \langle x_1^n \rangle \sim t^{-(n+2-\sqrt{n^2+4})/2}. \tag{5.49}$$

In particular, $\langle \ell \rangle \sim t^{-(3-\sqrt{5})/2} \sim t^{-.382}$ and $\langle \ell^2 \rangle^{1/2} \sim t^{-(2-\sqrt{2})/2} \sim t^{-.293}$. These moments decay slower than what one might naively anticipate from the behavior of the average area. From (5.48), the average area decays as $\langle A\rangle = \langle x_1x_2\rangle \sim t^{-1}$. Thus a natural length scale of the square-root of the area decays as $t^{-1/2}$. Another intriguing consequence of multiscaling is that a typical fragment becomes elongated (Fig. 5.5). We may quantify this visual asymmetry by the aspect ratio moments $\langle (x_1/x_2)^n\rangle$ which, from Eq. (5.48), diverge as $\langle (x_1/x_2)^n\rangle \sim t^{\sqrt{n^2+1}-1}$.

Scaling and Multiscaling

The notion of scaling, which permeates this book, typifies many complex systems. The notion of multiscaling is less well appreciated and it is worthwhile to give an example that illustrates the essential difference between scaling and multiscaling.

Consider the Gaussian probability distribution for a variable $x(t)$:

$$c(x,t) = \frac{1}{\sqrt{\pi t}}\, e^{-x^2/t}\ , \tag{5.50}$$

and let the exponent ν_n characterize the growth of the $n^{\rm th}$ moment of the variable x^2:

$$\langle x^{2n}\rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi t}}\, x^{2n}\, e^{-x^2/t}\, dx = \frac{t^n}{\sqrt{4\pi}}\,\Gamma\left(n+\tfrac{1}{2}\right).$$

Thus the exponent $\nu_n = n$. This linear dependent of the exponent on the order of the moment defines scaling behavior. Another way to express this fact is that the ratios

$$\frac{\langle x^{2n}\rangle}{\langle x^2\rangle^n}$$

do not depend on time. Thus the $n^{\rm th}$ moment of x^2 scales as the $n^{\rm th}$ power of $\langle x^2\rangle$. Thus a single time scale characterizes all the moments. Now consider the same Gaussian distribution for the variable $x = \ln y$. Using the variable transformation $c(y) = c(x)\frac{dx}{dy}$, we have

$$c(y,t) = \frac{1}{\sqrt{\pi t}}\,\frac{1}{y}\, e^{-(\ln y)^2/t}. \tag{5.51}$$

For this distribution, the $n^{\rm th}$ moment is

$$\begin{aligned}
\langle y^n\rangle &= \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-(\ln y)^2/t + n\ln y}\,\frac{dy}{y}\\
&= \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-(\ln y/\sqrt{t} - n\sqrt{t}/2)^2 + n^2t/4}\, d\ln y\\
&= \frac{1}{\sqrt{t}}\, e^{n^2t/4}.
\end{aligned}$$

The ratio $\langle y^n\rangle/\langle y\rangle^n$ grows rapidly with time and thus a single scale is insufficient to characterize the distribution of the variable $\ln y$. This lack of scaling is termed *multiscaling* or *multifractal scaling.*

Although the length and aspect ratio exhibit multiscaling, the area distribution obeys conventional single-parameter scaling in which, again from Eq. (5.48), the area moments are characterized by a linear exponent spectrum: $\langle A^n\rangle \sim t^{-n}$. The area distribution is derived from the multivariate distribution using $c(A,t) = \int\int c(x_1,x_2)\,\delta(x_1x_2 - A)\,dx_1\,dx_2$. The area distribution obeys a scaling form as in Eq. (5.18): $c(x_1,x_2) \simeq t^{-2}f(At)$ with the scaling function

$$f(z) = 6\int_0^1 d\xi\,(\xi^{-1}-1)e^{-z/\xi}. \tag{5.52}$$

The area distribution diverges weakly: $f(z) \simeq 6\ln\frac{1}{z}$ at small areas, $z \ll 1$. In the opposite limit, there is an exponential decay, reminiscent of the one-dimensional case, but with an algebraic correction: $f(z) \simeq 6z^{-2}\exp(-z)$ as $z \gg 1$. The ordinary scaling behavior of the area reflects the fact that the fragmentation rate equals the area.

Problems

5.1 Exact Solutions

1. Solve the random scission model using the scaling ansatz $c(x,t) \simeq t^2\Phi(xt)$ (see matters of technique).
2. Obtain the leading asymptotic behavior of the moments for the random scission model.

5.4 Rectangular Fragmentation

1. Determine the distribution of fragments of length x_1 and height x_2, $c(x_1,x_2)$, for the geometric fragmentation process illustrated below. Show that the two-variable Mellin transform $M(s_1,s_2)$ obeys

$$\frac{\partial M(s_1,s_2)}{\partial t} = \left(\frac{4}{s_1 s_2} - 1\right) M(s_1+1, s_2+1).$$

From this equation and using the assumption that $M(s_1,s_2) \sim t^{-\alpha(s_1,s_2)}$, show that

$$\alpha(n_1,n_2) = \left[(n_1+n_2) - \sqrt{(n_1-n_2)^2+16}\right]/2. \tag{5.53}$$