Appendix A

MATTERS OF TECHNIQUE

A.1 Transform Methods

Laplace transforms for continuum systems

Generating function for discrete systems

This method is demonstrated for the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2} \tag{A.1}$$

for $n \geq 1$ with $F_{-1} = 0$ and $F_0 = 1$. The first six terms in the sequence are $\{1, 1, 2, 3, 5, 8\}$. The generating function is defined as follows

$$F(z) = \sum_{n=0}^{\infty} F_n z^n. \tag{A.2}$$

To obtain the generating function, we multiply the recursion relation by z^n and sum over n. The left hand side yields $F(z) - 1$. The two terms on the right hand side yield $zF(z)$ and $z^2F(z)$, respectively. Thus, the generating function satisfies $F(z) = 1 + (z + z^2)F(z)$ and its solution is immediate

$$F(z) = \frac{1}{1 - z - z^2}. \tag{A.3}$$

We can manually check that this expression reproduces the first few sequence elements, $F(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$. The generating function is re-written as follows

$$F(z) = \frac{1}{1 - z - z^2} = \frac{1}{(1 - \lambda_+ z)(1 - \lambda_- z)} = \frac{1}{\lambda_+ - \lambda_-}\left[\frac{\lambda_+}{1 - \lambda_+ z} - \frac{\lambda_-}{1 - \lambda_- z}\right]. \tag{A.4}$$

Here, $\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$. Expanding the two fractions as Taylor series, the Fibonacci numbers follow

$$F_n = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}. \tag{A.5}$$

A.2 Relation between Laplace Transforms and Real Time Quantities

In many time-dependent phenomena, we want the asymptotic behavior of some quantity as a function of time when only its generating function or its Laplace transform is available. A typical example is a function $F(t)$ whose Laplace transform has the small-s behavior $F(s) \sim s^{\mu-1}$, or equivalently, whose generating function has the form $F(z) \sim (1 - z)^{\mu-1}$, with $\mu < 1$ as $z \to 1$ from below. We will show that the corresponding time dependence is $F(t) \sim t^{-\mu}$ as $t \to \infty$. The first point to make is that the generating function and the Laplace transform are equivalent. For a function $F(t)$ that is defined only for positive integer values, that is $t = n$, with n an integer, the Laplace transform is

$$F(s) = \int_0^\infty F(t)\, e^{-st}\, dt = \sum_{t=0}^{\infty} F(n)\, e^{-sn}.$$

Thus defining $z = e^{-s}$, the Laplace transform is the same as the generating function. In the limit $z \to 1$ from below, the sum cuts of only very slowly and can be replaced by an integral. This step leads to the Laplace transform in the limit $s \to 0$.

If we are interested only in long-time properties of a function $F(t)$, these features can be obtained through simple and intuitively appealing means. While lacking in rigor, this approach provides the correct behavior for all cases of physical interest. The most useful of these methods are outlined in this section. Let us first determine the long-time behavior of a function $F(t)$ when only its Laplace transform is known. There are two fundamentally different cases to consider: (a) $\int_0^\infty F(t)\, dt$ diverges or (b) $\int_0^\infty F(t)\, dt$ converges.

In the former case, relate the Laplace transform $F(s)$ to $F(t)$ by the following simple step:

$$F(s) = \int_0^\infty F(s)\, e^{-st}\, dt \approx \int^{t^*} F(t)\, dt. \tag{A.6}$$

That is, we simply replace the exponential cutoff in the integral, with characteristic lifetime $t^* = 1/s$, by a step function at t^*. Although this crude approximation introduces numerical errors of the order of 1, the essential asymptotic behavior of $F(t)$ is preserved when $\int 0^\infty F(t)\, dt$ diverges. Now if $F(t) \to t^{-\mu}$ with $\mu < 1$ as $t \to \infty$, then $F(s)$ diverges as

$$F(s) \sim \int^{1/s} t^{-\mu}\, dt \sim s^{\mu-1} \tag{A.7}$$

as $s \to 0$ from below. In summary, the fundamental connection when $\int_0^\infty F(t)\, dt$ diverges is

$$F(t) \sim t^{-\mu} \quad \longleftrightarrow \quad F(s) \sim s^{\mu-1}. \tag{A.8}$$

The above result also provides a general connection between the time integral of a function, $\mathcal{F}(t) \equiv \int_0^t F(t)\, dt$, and the Laplace transform of F. For $s = 1/t^*$ with $t^* \to \infty$, Eq. (A.7) is just the following statement:

$$F(s = 1/t^*) \sim \int_0^{t^*} F(t)\, dt = \mathcal{F}(t^*). \tag{A.9}$$

Thus a mere variable substitution provides an approximate, but asymptotically correct, algebraic relation between the Laplace transform of a function and the time integral of this same function. For this class of examples, there is no need to perform an integral to relate a function and its Laplace transform.

In the opposite situation where $\int_0^\infty F(t)\, dt = \mathcal{F}(\infty)$ converges, we can obtain the connection between $F(t)$ and $F(s)$ in a slightly different way. Let us again suppose that $F(t) \sim t^{-\mu}$ as $t \to \infty$, but now with $\mu > 1$ so that $F(s)$ is finite as $s \to 0$. Exploiting the fact that the time integral of $F(t)$ converges, we write

$$\begin{aligned} F(s) &= \int^\infty t^{-\mu} \left[1 - (1 - e^{-st})\right] dt, \\ &\sim \mathcal{F}(\infty) + \int_{1/s}^\infty t^{-\mu}\, dt, \\ &\sim \mathcal{F}(\infty) + s^{\mu-1}. \end{aligned} \tag{A.10}$$

Again, we replace the exponential cutoff in the integrand by a sharp cutoff.

In summary, the small-s behavior of the Laplace transform, or, equivalently, the $z \to 1$ behavior of the generating function, are sufficient to determine the long-time behavior of the function itself. Since the transformed quantities are usually easy to obtain by the solution of an appropriate boundary-value problem, the asymptotic methods outlined here provide a simple route to obtain long-time behavior.

In the context of time-dependent phenomena, one of the most useful features of the Laplace transform is that it encodes all positive integer powers of the mean time. That is, we define the positive integer moments of $F(t)$ as

$$\langle t^n \rangle = \frac{\int_0^\infty t^n \, F(t) \, dt}{\int_0^\infty F(t) \, dt}. \tag{A.11}$$

If all these moments exist, then $F(s)$ can be written as a Taylor series in s These generate the positive integer moments of $F(t)$ by means of

$$\begin{aligned} F(s) &= \int_0^\infty F(t) \, e^{-st} \, dt \\ &= \int_0^\infty F(t) \left(1 - st + \frac{s^2 t^2}{2!} - \frac{s^3 t^3}{3!} + \dots \right) \\ &= \mathcal{F}(\infty) \left(1 - s\langle t \rangle + \frac{s^2}{2!}\langle t^2 \rangle - \frac{s^3}{3!}\langle t^3 \rangle + \dots \right). \end{aligned} \tag{A.12}$$

Thus the Laplace transform is a *moment generating function*, as it contains *all* the positive integer moments of the probability distribution $F(t)$.

A.3 Asymptotic Analysis

This method is demonstrated for the Fibonacci numbers, defined according to the recursion rule (A.1). How do these numbers grow with n? The simpler series $F_n = 2F_{n-1}$ with $F_0 = 1$ is simply the geometric series $F_n = 2^n$. Thus, we try

$$F_n \sim \lambda^n \tag{A.13}$$

and expect $z < 2$. Substituting this ansatz into the recurrence relation (A.1) shows that the ansatz is compatible with the equation when $\lambda^2 = \lambda + 1$. There are two roots $\lambda_\pm = \left(1 \pm \sqrt{5}\right)/2$. The largest root is the relevant one asymptotically, $\lambda = \lambda_+$. This number, the golden ratio, characterizes the growth of the sequence

$$\lambda = \frac{1+\sqrt{5}}{2}. \tag{A.14}$$

Thus, the growth of the sequence is obtained up to the proportionality coefficient $\lim_{n\to\infty} F_n/\lambda^n$. This prefactor is of secondary importance, and it is of the order unity. In this case, an exact solution of F_n was possible, but in typical applications, exact solutions are cumbersome and asymptotic analysis are quite useful.

Steepest Descent

A.4 Scaling Approaches

Separation of variables

Conversion of PDEs to ODEs

A.5 Differential Equations

We demonstrate several useful methods for the same problem: aggregation with constant kernel. The Smoluchowsky equation) for c_k, the density of clusters of size k is

$$\frac{dc_k}{dt} = \sum_{i+j=k} c_i c_j - 2Nc_k \tag{A.15}$$

and the initial conditions $c_k(0) = \delta_{k,0}$.

Moments methods

The moments are defined as follows

$$M_n(t) = \sum_k x^n c_k. \tag{A.16}$$

Multiplying the rate equations by k^n and summing over k, the moments obey the closed set of equations

$$\frac{dM_n}{dt} = \sum_{l=0}^{n} \binom{n}{l} M_l M_{n-l} - 2M_0 M_n. \tag{A.17}$$

Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients. The zeroth moment satisfies $\frac{dM_0}{dt} = -M_0^2$ and the initial conditions $M_0 = 1$, so the solution is $M_0 = (1+t)^{-1}$. The first moment (total mass) is conserved, $M_1 = 1$.

The rest of the moments can be solved recursively. However, such an approach may not be practical. Instead, we solve for the leading behavior of the moments in the long time limit. We assume that average mass, $\langle k \rangle = M_1/M_0$, characterizes all moments, $M_n \sim M_0(M_1/M_0)^n$, with the normalization assuring that the zeroth order moment is recovered. Since asymptotically $\langle k \rangle \simeq t$, the scaling ansatz is

$$M_n \simeq A_n\, t^{n-1} \tag{A.18}$$

with $A_0 = A_1 = 1$. Substituting this ansatz into the moment equation, these coefficients satisfy the recursion relation

$$(n+1)\frac{A_n}{n!} = \sum_{l+m=n} \frac{A_l}{l!}\frac{A_m}{m!}. \tag{A.19}$$

To solve this equation we introduce the generating function

$$A(z) = \sum_n \frac{A_n}{n!} z^n. \tag{A.20}$$

Given the structure of the recursion relation, we conveniently absorbed the factor $n!$ into the definition of the generating function. Multiplying the recursion relation by z^n and summing over n, the generating function satisfies

$$\frac{d}{dz}\left[zA(z)\right] = A^2(z). \tag{A.21}$$

and the boundary condition $A(0) = 0$. This equation can be re-written as the Ricatti equation $zA_z + A = A^2$. Thus, we introduce $g(z) = A^{-1}(z)$ which satisfies $zg_z - g = -1$. The solution to this equation subject to $g(0) = 1$ is $g(z) = 1 - z$. Therefore,

$$A(z) = \frac{1}{1-z}. \tag{A.22}$$

This yields the coefficients $A_n = n!$ and therefore, the leading asymptotic behavior of the moments

$$M_n \simeq n! t^{n-1}. \tag{A.23}$$

The asymptotic form of the size distribution can be obtained from the asymptotic form of the moments. Since the scale $k \sim t$ characterizes the moments, the size distribution attains the form $c_k \sim t^{-2}\Phi(kt^{-1})$. This form is consistent with the moment behavior with the corresponding coefficients merely the moments of the scaling distribution $A_n = \int dx x^n \Phi(x)$. In this case, the coefficients are of a simple form and thus allow us to guess the distribution: $n! = \int dx x^n \exp(-x)$ so $\Phi(x) = \exp(-x)$.

Scaling Solutions

Usually it is much simpler to solve for the scaling function rather than seek a complete exact solution. The exact moment solutions $M_0 = (1+t)^{-1}$ and $M_1 = 1$ show that the average size grows linearly with time $\langle k \rangle = 1 + t \simeq t$. The scaling ansatz

$$c_k \simeq t^{-2}\Phi(kt^{-1}) \tag{A.24}$$

is consistent with the first two moments when $\int dz\, \Phi(z) = \int dz\, z\, \Phi(z) = 1$. Replacing the sum in the rate equation by an integral and substituting this scaling form leads to the following integro-differential equation

$$-z\frac{d}{dz}\Phi(z) = \int_0^z dy\, \Phi(x)\Phi(z-x). \tag{A.25}$$

The convolution structure suggests the Laplace transform: $\hat{\Phi}(s) = \int dz\, e^{-sz}\Phi(z)$ with the normalization dictating $\hat{\Phi}(s)|_{s=0} = 1$ and $\frac{d}{ds}\hat{\Phi}(s)|_{s=0} = -1$. The Laplace transform obeys

$$\frac{d}{ds}\left[s\,\hat{\Phi}(s)\right] = \hat{\Phi}^2(s). \tag{A.26}$$

This Ricatti equation is linearize via the transformation $g = 1/\hat{\Phi}$: $-s\frac{d}{ds}g + g = 1$. With the boundary conditions $g(s)|_{s=0} = \frac{d}{ds}g(s)|_{s=0} = 1$ the solution is $g(s) = 1+s$ and therefore, $\hat{\Phi}(s) = (1+s)^{-1}$. The inverse Laplace transform is the exponential

$$\Phi(x) = \exp(-x). \tag{A.27}$$

A.6 Partial Differential Equation

Traveling Waves & Extremum Selection

In many situations, a solution to a partial differential equation attains a traveling wave form asymptotically, thereby reducing the problem complexity. We describe a handy technique for obtaining the propagation velocity of the traveling wave.

Consider the classic Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) population equation

$$u_t = u_{xx} + u(1-u). \tag{A.28}$$

This partial differential equation describes changes in the average population $u(x,t)$ at position x at time t due to diffusion, birth (linear term), and death as a result of overpopulation (non-linear term). Furthermore, consider the step initial condition $u_0(x) = \Theta(x)$ with $\Theta(x)$ the Heaviside step function: $\Theta(x) = 1$ for $x < 0$ and $\Theta(x) = 0$ for $x > 0$.

There are two steady state solutions: an unstable solution, $u = 0$, and a stable solution, $u = 1$. One anticipates that the stable phase penetrates the unstable one. Furthermore, assume that asymptotically (as $t \to \infty$) the stable phase propagates into the unstable one with a constant velocity c

$$u(x,t) \to F(x-ct). \tag{A.29}$$

The wave-function $F(z)$ satisfies the ordinary differential equation

$$F_{zz} + c\,F_z + F(1-F) = 0 \tag{A.30}$$

and the boundary conditions $F(-\infty) = 1$ and $F(\infty) = 0$. The wave front is located at $x \approx c\,t$. Far ahead of this position, the population is very sparse, so the nonlinear term is negligible. The resulting linear equation $F_{zz} + cF_z + F = 0$ implies an exponential front, $F(z) \propto \exp(-\nu z)$ with the decay coefficient ν satisfying $\nu^2 - c\nu + 1 = 0$. Alternatively, the propagation velocity and the decay coefficient are related via

$$c = \frac{\nu^2+1}{\nu}. \tag{A.31}$$

This curve has a minimum at $c_{\rm min} = 2$, realized when $\nu = \nu_{\rm min} = 1$. Even though there is a continuous family of solutions, characterized by the velocity c, the minimal value

$$c = 2 \tag{A.32}$$

is actually selected by the dynamics[1]!

This "extremum selection principle" is ubiquitous and very handy. It yields the propagation velocity and reduces the partial differential equation (A.28) into the ordinary differential equation (A.30). This is a considerable simplification from a dependence on two variable x and t to a dependence on only one $z = x-ct$, a transformation equivalent to a scaling transformation. In its core, though it is a *linear* analysis method.

Special solutions in the family of solutions can be still realized. When the initial condition is not compact, but rather has the extended tail, $u_0(x) \sim \exp(-\nu z)$, then the velocity is given by (A.31). This selection is sometimes termed "weak selection".

A.7 Extreme Statistics

Extremes of simple distributions

Lifshitz tail argument

Combining an exact solution of a sub-class of extreme events with the scaling behavior results in a powerful heuristic tool for characterizing extremal statistics.

[1]This can be justified rigorously using a saddle point analysis in the complex plane of a wave dispersion relation equivalent to (A.31).

Consider the discrete time random walk. At time t, the walker is at position $x(t)$, defined recursively via

$$x(t+1) = \begin{cases} x(t) - 1 & \text{with probability } 1/2; \\ x(t) + 1 & \text{with probability } 1/2. \end{cases} \tag{A.33}$$

The walker starts at the origin $x(0) = 0$. The average displacement satisfies the recursion $\langle x(t+1)\rangle = \langle x(t)\rangle + \frac{1}{2}\times(-1) + \frac{1}{2}\times(1) = \langle x(t)\rangle$ and therefore, the average vanishes $\langle x(t)\rangle = 0$. The variance in the displacement satisfies the recursion $\langle x^2(t+1)\rangle = \frac{1}{2}\langle (x(t)-1)^2\rangle + \frac{1}{2}\langle (x(t)+1)^2\rangle = \langle x^2(t)\rangle + 1$. Therefore, the variance grows linearly in time $\langle x^2(t)\rangle = t$, a diffusive behavior. Given the growth $x \simeq \sqrt{t}$ we assume that $P(x,t)$, the probability that the walker is at position x at time t, becomes self-similar

$$P(x,t) \simeq \frac{1}{\sqrt{t}}\Phi\left(\frac{x}{\sqrt{t}}\right). \tag{A.34}$$

An easy to analyze sub-class of extreme random walkers are those that always take a positive step. These walkers move ballistically, $x = t$, and their likelihood is $P(x = t, t) = 2^{-t}$. This probability decreases exponentially with time

$$P(x = t, t) \sim \exp(-Ct). \tag{A.35}$$

Matching the scaling behavior (A.34) with the tail (A.35) shows that the scaling function satisfies $\Phi(\sqrt{t}) \sim \exp(-C\,t)$ (Here, we ignored the secondary algebraic correction $1/\sqrt{t}$.). This yields the tail behavior

$$\Phi(z) \sim \exp(-Cz^2) \tag{A.36}$$

as $z \to \infty$. Thus, we have obtained the functional form of the tail. (This approach does not produce the proportionality constant C.) The moral is that seemingly trivial and partial description, in this case, ballistic walkers, can yield via proper manipulation useful generic information.

Reversion of Series

A.8 Probability theory

Generating functions

Moments and cumulants

Elementary distributions: Normal & Poisson

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Appendix B

Formulas & Distributions

B.1 Useful Formulas

- The Gamma function
$$\Gamma(n) = \int_0^\infty dx\, x^{n-1} e^{-x}. \tag{B.1}$$

  1. Large-n asymptotics (the Stirling formula)
  $$\Gamma(n) \simeq \sqrt{2\pi n}\, n^n e^{-n}. \tag{B.2}$$

  2. Recursion relation
  $$\Gamma(n+1) = n\Gamma(n) \tag{B.3}$$

  3. Asymptotic ratio
  $$\Gamma(x+a)/\Gamma(x) \simeq x^a \tag{B.4}$$

- The incomplete Gamma function
$$\Gamma(n, y) = \int_y^\infty dx\, x^{n-1} e^{-x}. \tag{B.5}$$

- The exponential integral
$$\mathrm{Ei}\,(x) = \int_x^\infty du \frac{e^{-u}}{u}. \tag{B.6}$$

  1. Asymptotic behavior
  $$\mathrm{Ei}\,(x) = -\ln x - \gamma \tag{B.7}$$

- The Beta function
$$B(n, m) = \int_0^1 dx\, x^{n-1}(1-x)^{m-1}. \tag{B.8}$$

  1. Relation to Gamma function
  $$B(n, m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \tag{B.9}$$