(1)

$$\frac{\partial c_k}{\partial t} = D \frac{\partial^2 c_k}{\partial x_k^2}$$

c_k = concentration in k^{th} branch

x_k = position on k^{th} branch.

$k = 1, 2, 3$

a) $c_k(x_k = L_k, t) = 0$

b) $c_1(x_1 = 0, t) = c_2(x_2 = 0, t) = c_3(x_3 = 0, t)$

c) $\sum_{k=1}^{3} j_k(x_k = 0, t) = \delta(t)$, $j_k = -D \frac{\partial c_k}{\partial x_k}$ = flux

i.e. after $t=0$, incoming flux equals outgoing flux at the vertex

General sol'n in Laplace domain is:

$$c_k(x_k, s) = A_k \sinh\left[\sqrt{\tfrac{s}{D}}\,(L_k - x_k)\right]$$

condition (b) gives:

$$A_1 \sinh\left(\sqrt{\tfrac{s}{D}}\, L_1\right) = A_2 \sinh\left(\sqrt{\tfrac{s}{D}}\, L_2\right) = A_3 \sinh\left(\sqrt{s/D}\, L_3\right)$$

since $c_1 = A_1 \sinh \sqrt{\tfrac{s}{D}}\, L_1$,

$$c_2 = A_1 \frac{\sinh \sqrt{s/D}\, L_1}{\sinh \sqrt{s/D}\, L_2} \sinh\left[\sqrt{s/D}\,(L_2 - x_2)\right]$$

$$c_3 = A_1 \frac{\sinh \sqrt{s/D}\, L_1}{\sinh \sqrt{s/D}\, L_3} \sinh\left[\sqrt{s/D}\,(L_3 - x_3)\right]$$

②

so

$$j_1 = \sqrt{sD}\; A_1 \cosh\left[\sqrt{s/D}\,(L_1 - x_1)\right]$$

$$j_2 = \sqrt{sD}\; A_2 \frac{\sinh \sqrt{s/D}\,L_1}{\sinh \sqrt{s/D}\,L_2} \cosh\left[\sqrt{s/D}\,(L_2 - x_2)\right]$$

$$j_3 = \sqrt{sD}\; A_3 \frac{\sinh \sqrt{s/D}\,L_1}{\sinh \sqrt{s/D}\,L_3} \cosh\left[\sqrt{s/D}\,(L_3 - x_3)\right]$$

Then since $\sum j_K = 1$,

$$1 = \sqrt{sD}\; A_1 \Big[\cosh \sqrt{s/D}\,L_1 + \frac{\sinh \sqrt{s/D}\,L_1}{\sinh \sqrt{s/D}\,L_2} \cosh \sqrt{s/D}\,L_2 + \frac{\sinh \sqrt{s/D}\,L_1}{\sinh \sqrt{s/D}\,L_3} \cosh \sqrt{s/D}\,L_3 \Big]$$

so

$$A_1 = \frac{1}{\sqrt{sD}} \left\{ \cosh\left(\sqrt{s/D}\,L_1\right) + \sinh\left(\sqrt{s/D}\,L_1\right) \left[\coth \sqrt{s/D}\,L_2 + \coth \sqrt{s/D}\,L_3\right] \right\}^{-1}$$

giving

$$j_1 = \frac{1}{\cosh \sqrt{s/D}\,L_1 + \sinh \sqrt{s/D}\,L_1 \left(\coth \sqrt{s/D}\,L_2 + \coth \sqrt{s/D}\,L_3\right)}$$

$$j_2 = \frac{\sinh \sqrt{s/D}\,L_1 / \sinh \sqrt{s/D}\,L_2}{\cosh + \sinh\,(\coth + \coth)}$$

or

$$j_K = \frac{\operatorname{cosech} \sqrt{s/D}\,L_K}{\coth \sqrt{s/D}\,L_1 + \coth \sqrt{s/D}\,L_2 + \coth \sqrt{s/D}\,L_3}$$

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The exit probability is the $s \to 0$ limit of these rates:

$$\operatorname{cosech} \sqrt{s/D}\, L \sim \coth \sqrt{s/D}\, L = \frac{\sqrt{D/s}}{L}$$

so $$P^{exit}_K = \frac{\sqrt{\frac{D}{s}}\, L_K^{-1}}{\sum \sqrt{\frac{D}{s}}\, L_K^{-1}} = \frac{L_K^{-1}}{\sum L_K^{-1}}$$

a) $$P^{ex}_1 = \frac{1/L_1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}} = \frac{\frac{1}{L_1}}{\frac{L_1 + L_2 + L_3}{L_1 L_2 L_3}} = \frac{L_2 L_3}{L_1 + L_2 + L_3}$$

$$P^{ex}_2 = \frac{L_1 L_3}{L_1 + L_2 + L_3}, \quad P^{ex}_3 = \frac{L_1 L_2}{L_1 + L_2 + L_3}$$

b) The average absorption time is the first correction term in an expansion of the flux:

$$j_K \approx \frac{\frac{L_1}{L_K}\left(\frac{1 + sL_1^2/6D}{1 + sL_K^2/6D}\right)}{\left(1 + \frac{sL_1^2}{2D}\right) + \sqrt{s/D}\, L_1\left(1 + \frac{sL_1^2}{6D}\right)\left[\frac{1}{\sqrt{s/D}\, L_2}\left(1 + \frac{sL_2^2}{3D}\right) + \frac{1}{\sqrt{s/D}\, L_3}\left(1 + \frac{sL_3^2}{3D}\right)\right]}$$

$$\approx \frac{\frac{1}{L_K}\left(1 + s(L_1^2 - L_K^2)/6D\right)}{M_{-1} + \frac{s}{D}\left[\frac{L_1^2}{2} + \left(\frac{L_1^3}{6L_2} + \frac{L_1 L_2}{3}\right) + \left(\frac{L_1^3}{6L_3} + \frac{L_1 L_3}{3}\right)\right]}$$

$-M_{-1} = \sum_K \frac{1}{L_K}$

$-M_1 = \sum_K L_K$

(4)

$$j_K \simeq \frac{1/L_K \left(1 + s(L_1^2 - L_K^2)/6D\right)}{M_{-1} + \frac{s}{D}\left(\frac{L_1^2}{6} M_{-1} + \frac{1}{3} M_1\right)}$$

$$\simeq \frac{1/L_K}{M_{-1}}\left[1 - \frac{s}{6D}\left(L_K^2 + \frac{2M_1}{M_{-1}}\right)\right]$$

so $$\langle t_K \rangle = \frac{L_K^2}{6D} + \frac{1}{3D}\frac{M_1}{M_{-1}}$$

or

$$\langle t_1 \rangle = \frac{L_1^2}{6D} + \frac{L_1 L_2 L_3}{3D}$$

$$\langle t_2 \rangle = \frac{L_2^2}{6D} + \frac{L_1 L_2 L_3}{3D}$$

$$\langle t_3 \rangle = \frac{L_3^2}{6D} + \frac{L_1 L_2 L_3}{3D}$$

b) $$\langle t \rangle = \sum_K \langle t_K \rangle P_K$$

$$= \sum_K \frac{L_K^{-1}}{M_{-1}}\left(\frac{L_K^2}{6D} + \frac{M_1}{3D M_{-1}}\right)$$

$$= \sum_K \frac{L_K}{6D M_{-1}} + \frac{L_K^{-1} M_1}{3D M_{-1}^2}$$

$$= \frac{L_1 + L_2 + L_3}{6D M_{-1}} + \frac{M_1 M_{-1}}{3D M_{-1}^2} = \frac{M_1}{6D M_{-1}} + \frac{M_1}{3D M_{-1}}$$

$$= \frac{M_1}{3D M_{-1}}\left(\frac{3}{2}\right) = \boxed{\frac{L_1 L_2 L_3}{2D} = \langle t \rangle}$$

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2.

start with the most general exponential form

$$f(u) = a e^{-bu}$$

plug into equation:

$$2f(u) + u f'(u) + c_2 K(u) = 0 \qquad \text{where}$$

$$K(u) = \frac{1}{2}\int_0^u dv\, c_1 f(v) f(u-v) - \int_0^\infty dv\, c_1 f(u) f(v)$$

$$K(u) = \frac{1}{2}\int_0^u dv\, c_1 a^2 \cancel{e^{-bv}} e^{-bu} \cancel{e^{bv}} - \int_0^\infty dv\, c_1 a^2 e^{-b(u+u)}$$

$$= c_1 a^2 e^{-bu} u \quad - \quad c_1 a^2 e^{-bu} \cancel{\int_0^\infty e^{-bv}} \frac{1}{b}$$

$$K(u) = c_1 a^2 e^{-bu}\left[\frac{u}{2} - \frac{1}{b}\right]$$

plugging into main equation:

$$\left(2 - \frac{c_2 c_1 a^2}{b}\right) + u\left(\frac{c_2 c_1 a^2}{2} - b\right) = 0$$

so $$b = \frac{c_2 c_1 a^2}{2}$$

for simplicity, we pick $a = 1$ $\quad c_1 = 2 \quad \Rightarrow \quad b = c_2$

$$\therefore f(u) = e^{-\frac{u}{\Lambda}}$$

(6)

3

product kernel $(K = i \times j)$ with input

a) $$\frac{\partial c_k}{\partial t} = \frac{1}{2} \sum_{i+j=k} i j c_i c_j - \sum_i i k c_i c_k + \delta_{1,k}$$

loss gain input.

$$M_2(t) = \sum_k k^2 c_k(t)$$

multiply both sides of eqn a by $\sum_k k^n$

$$\dot{M}_2 = \frac{1}{2} \sum_{i,j} (i+j)^2 i j c_i c_j - \sum_{i,k} i k^3 c_i c_k + 1$$

$$= \frac{1}{2} \sum_{i,j} \left(i^3 j + j^3 i + 2 i^2 j^2 \right) c_i c_j - \sum_{i,k} i k^3 c_i c_k + 1$$

$$= \frac{1}{2} \left(2 M_3 M_1 + 2 M_2^2 \right) - M_1 M_3 + 1$$

$$= \dot{M}_2 = M_2^2 + 1 \Rightarrow \frac{dM_2}{1 + M_2} = t$$

$$\Rightarrow \tan^{-1} M_2(t) - \tan^{-1} M_2(0) = t$$

setting $M_2(0) = 0$ (no clusters of any size at $t=0$)

$$\boxed{M_2(t) = \tan t}$$

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find $N = M_0$ using same method as in a)

$$\dot{M}_0 = \frac{1}{2}(M_1^2) - M_1^2 + 1 = 1 - \frac{1}{2}M_1^2$$

$$\dot{M}_1 = \frac{1}{2}(M_2M_1 + M_2M_1) - M_2M_1 + 1 = 1$$

$$M_1(t) = t$$

plugging back in:

$$\dot{M}_0 = 1 - \frac{1}{2}t^2 \Rightarrow \boxed{M_0 = t - \frac{t^3}{6}}$$