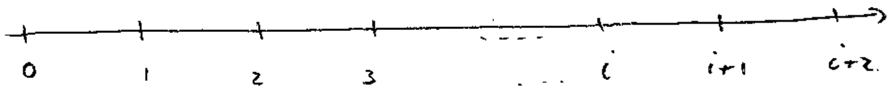
1



$$W(\sigma_n) = \frac{1}{2}\left(1 - \frac{\sigma_n}{2}\left(\sigma_{n+1} + \sigma_{n+2}\right)\right)$$

$$= \frac{1}{2} - \frac{\sigma_0\sigma_1}{4} - \frac{\sigma_0\sigma_2}{4}$$

with $s_n \equiv \langle \sigma_n \rangle$ $\quad S_n(t=0) = \delta_{n,0}$

$$\dot{S}_n = -S_n + \frac{1}{2}\left(S_{n+1} + S_{n+2}\right)$$

let $\quad S(k) \equiv \sum_{n=-\infty}^{\infty} s_n e^{ikn}$

$$\dot{S}(k) = -\left(1 - \frac{1}{2}\left(e^{-ik} + e^{-2ik}\right)\right) S(k)$$

rewrite in terms of sin's + cos's solve for $S(k)$ using $S(k, t=0) = 1$

$$S(k) = e^{-t\left(1 - \frac{1}{2}(\cos k + \cos 2k) + \frac{i}{2}(\sin k + \sin 2k)\right)}$$

use the following definitions: $\quad e^{z\cos k} = \sum_{n=-\infty}^{\infty} e^{ikn} I_n(z)$

$$e^{z \sin k} = \sum_{n=-\infty}^{\infty} e^{ikn}(-i)^n I_n(z)$$

$$S(k) = e^{-t}\left(\sum_{a=-\infty}^{\infty} e^{ika} I_a(t/2)\right)\left(\sum_{b=-\infty}^{\infty} e^{ikb} I_b(t/2)(-i)^b\right) \times$$

$$\times \left(\sum_{c=-\infty}^{\infty} e^{ik2c} I_c(t/2)\right)\left(\sum_{d=-\infty}^{\infty} e^{ik2d} I_d(t/2)(-i)^d\right)$$

$$= e^{-t}\sum_{a,b,c,d} e^{ik(a+b+2c+2d)}(-i)^{b+d} I_a I_b I_c I_d$$

1 cont

* Chris Sermo's idea:

let $a+b+2c+2d = n$

$$S(k) = e^{-t} \sum_{n=-\infty}^{\infty} e^{ikn} \left(\sum_{bcd} (-i)^{b+d} I_{n-b-2c-2d} I_b I_c I_d \right)$$

$$\therefore \boxed{S_n(t) = e^{-t} \sum_{b,c,d=-\infty}^{\infty} (-i)^{j+m} I_{n-b-2c-2d}\left(\tfrac{t}{2}\right) I_b\left(\tfrac{t}{2}\right) I_c\left(\tfrac{t}{2}\right) I_d\left(\tfrac{t}{2}\right)}$$

#2

$$\dot{v} = -\gamma v + \eta(t)$$

$$v(t) = e^{-\gamma t}\int_0^t e^{\gamma t'}\eta(t')\,dt' + v(0)e^{-\gamma t}$$

$$\langle v(t)\rangle = v(0)e^{-\gamma t} \qquad \langle v(t)\rangle\Big|_{t\to\infty} = \frac{kT}{m}$$

$$x(t) = v(0)\left(\frac{1-e^{-\gamma t}}{\gamma}\right) + \int_0^t\int_0^t dt\,dt'\,e^{-\gamma(t-t')}\eta(t')$$

$$= v(0)\left(\frac{1-e^{-\gamma t}}{\gamma}\right) + \int_0^t dt'\,\frac{1-e^{-\gamma(t-t')}}{\gamma}\,\eta(t')$$

then, for $\langle \eta(t)\rangle = 0$

$$\langle x(t)\rangle = v(0)\,\frac{1-e^{-\gamma t}}{\gamma}$$

$\langle x^2\rangle$:

$$\langle x(t)^2\rangle = \left[v(0)\left(\frac{1-e^{-\gamma t}}{\gamma}\right)\right]^2 + \cancel{\left[v(0)\left(\frac{1-e^{-\gamma t}}{\gamma}\right)\right]\int_0^t dt'\,\frac{1-e^{-\gamma(t-t')}}{\gamma}\langle\eta(t')\rangle} \to 0$$

$$\int_0^t\int_0^t dt'\,dt''\,\frac{1}{\gamma}\left(1-e^{-\gamma(t-t')}\right)\frac{1}{\gamma}\left(1-e^{-\gamma(t-t'')}\right)\langle\eta(t')\eta(t'')\rangle$$

$$= \frac{\Gamma}{\gamma^2}\int_0^t dt'\left(1-e^{-\gamma(t-t')}\right)^2 = \frac{\Gamma}{\gamma^2}\left(t - \frac{2}{\gamma}\left(1-e^{-\gamma t}\right) + \frac{1}{2\gamma}\left(1-e^{-2\gamma t}\right)\right)$$

#2 cont.

using $v(0)^2 = \frac{kT}{m}$ $\Gamma = \frac{kT\,2\gamma}{m}$

$$\langle x(t)^2\rangle = \left(v(0)\left(\frac{1-e^{-\gamma t}}{\gamma}\right)\right)^2 + \frac{\Gamma}{\gamma^2}\left(t - \frac{2}{\gamma}(1-e^{-\gamma t}) + \frac{1}{2\gamma}(1-e^{-2\gamma t})\right)$$

$$\langle x(t)^2\rangle = \frac{kT}{m\gamma^2}\left(2t\gamma - 4(1-e^{-\gamma t}) + 1 - \cancel{e^{-2\gamma t}}\right)$$

$$+ \frac{kT}{m\gamma^2}\left(1 - 2e^{-\gamma t} + \cancel{e^{-2\gamma t}}\right)$$

$$\boxed{\langle x(t)^2\rangle = \frac{2kT}{m\gamma^2}\left(t - \frac{1}{\gamma}(1-e^{-\gamma t})\right)}$$

\# 3 (soln by Chris Sarmo. Another version in Coffey + Morita J. Phys. D Vol 9, 1976)

$$\ddot{\phi} + \gamma\dot{\phi} = L(t) \qquad \langle L(t)\rangle = 0$$

$$\langle L(t)L(t')\rangle = \Gamma\,\delta(t-t')$$

Let $\theta = \dot{\phi}$

$$\dot{\theta} + \gamma\theta = L(t) \quad \leftarrow \text{eqn of motion for brownian particle.}$$

$$\dot{(\theta e^{\gamma t})} = L(t)e^{\gamma t}$$

$$\Rightarrow \theta(t) = \theta(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t dt' L(t')e^{\gamma t'}$$

pick $\phi(0) = 0 \Rightarrow \phi(t) = \int_0^t dt'\,\theta(t')$

$\langle \cos\phi(t_1)\cos\phi(t_2)\rangle =$ assuming noise dep on $|t_1 - t_2|$

$$= \langle \cos\phi(t_2-t_1)\cancelto{1}{\cos\phi(t=0)}\rangle = \langle\cos\phi(t_2-t_1)\rangle$$

$$\langle\cos\phi(\Delta t)\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\langle(\phi(\Delta t))^{2n}\rangle$$

assume white Gausian noise

$$\langle(\phi(\Delta t))^{2n}\rangle = \frac{(2n)!}{2^n n!}\langle(\phi(\Delta t))^2\rangle^n$$

$$\langle\cos\phi(\Delta t)\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\langle(\phi(\Delta t))^2\rangle^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{(-1)\langle(\phi(\Delta t))^2\rangle}{2}\right)^n$$

$$= e^{-\langle(\phi(\Delta t))^2\rangle/2}$$

#3 cont

where $\langle (\phi(\Delta t))^2 \rangle = \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \,\theta(t')\,\theta(t'')$

$$= \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \left(\theta(0) e^{-\gamma t'} + e^{-\gamma t'} \int_0^{t'} d\tau' L(\tau') e^{\gamma \tau'}\right) \times$$

$$\times \left(\theta(0) e^{-\gamma t''} + e^{-\gamma t''} \int_0^{t''} d\tau'' L(\tau'') e^{\gamma \tau''}\right)$$

this calculation is solved in Problem #2 if we note that

$\theta(0) = \dot{\phi}(t)\big|_{t=0} = w(0)$ $\qquad w \equiv$ angular speed.

$$\langle \phi(\Delta t)^2 \rangle = \frac{w(0)^2}{\gamma^2}\left(1 - 2e^{-\gamma t} + e^{-2\gamma t}\right) + \frac{kT}{\gamma^2 m}\left(2\gamma t + 4e^{-\gamma t} - 2e^{-2\gamma t} - 3\right)$$

$\langle \cos\phi(t_1) \cos\phi(t_2) \rangle =$

$$\exp\left\{ -\frac{1}{2}\left(\frac{w(0)^2}{\gamma^2}\left(1 - 2e^{-\gamma \Delta t} + e^{-2\gamma \Delta t}\right) + \frac{kT}{\gamma^2 m}\left(2\gamma \Delta t + 4e^{-\gamma \Delta t} - 2e^{-2\gamma \Delta t} - 3\right)\right)\right.$$

$\Delta t = |t_2 - t_1|$

$w(0) \equiv$ initial angular speed.

#4

$$\partial_t h(\vec{x},t) = \nabla^2 h(\vec{x},t) + \eta(\vec{x},t)$$

$$\langle \eta(\vec{x},t)\rangle = 0 \qquad \langle \eta(\vec{x}',t')\cdot\eta(\vec{x},t)\rangle = \delta^d(\vec{x}-\vec{x}')\delta(t-t')$$

Assume initial conditions $h(\vec{x},0)=0$
+ contour conditions $h(\vec{R},t)=0$

F.T. of both time and space

$$i\omega h(\vec{k},\omega) = -k^2 h(\vec{k},\omega) + \eta(\vec{k},\omega)$$

$$h(\vec{k},\omega) = \frac{\eta(\vec{k},\omega)}{k^2+i\omega}$$

$$\langle \eta(\vec{k},\omega)\cdot\eta(\vec{k}',\omega')\rangle = \int d^dx\, d^dx'\, dt\, dt'\, e^{i\vec{k}\vec{x}+i\vec{k}'\vec{x}'-i\omega t-i\omega' t'} \langle \eta(\vec{x},t)\cdot\eta(\vec{x}',t')\rangle$$

$$= (2\pi)^{d+1}\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')$$

$$\langle h(\vec{x},t)\cdot h(\vec{x}',t')\rangle = \int \frac{d^dk\, d^dk'\, d\omega\, d\omega'}{(2\pi)^{2(d+1)}} e^{-i\vec{k}\vec{x}-i\vec{k}'\vec{x}'-i\omega t-i\omega' t'} \cdot \frac{\langle \eta(\vec{k},\omega)\eta(\vec{k}',\omega')\rangle}{(k^2+i\omega)(k^2-i\omega)}$$

$$= \frac{1}{(2\pi)^d}\int d^dk \cos\vec{k}(\vec{x}-\vec{x}')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\cos\omega(t-t')}{(k^2+i\omega)(k^2-i\omega)}$$

Integrate in frequency domain

$$\frac{1}{(2\pi)^d}\int d^dk\, e^{-i\vec{k}(\vec{x}-\vec{x}')}\frac{e^{-k^2|t-t'|}}{2k^2}$$

trick: can write $\cos\square$ or $e^{\square}$ b/c imaginary part vanishes

Divide $\vec{k}$ into parallel $\vec{k}_\parallel$ and perpendicular $\vec{k}_\perp$

$$\frac{1}{2(2\pi)^d}\int d^{d-1}k_\perp\, e^{-k_\perp^2|t-t'|}\int_{-\infty}^{\infty} dk_\parallel \frac{\cos(k_\parallel|\vec{x}-\vec{x}'|)}{k_\parallel^2+k_\perp^2} e^{-k_\parallel^2|t-t'|}$$

#4 cont.

$$= \frac{1}{2(2\pi)^d} \int d^{(d-1)} k_\perp \, e^{-k_\perp^2 |t-t'|} \frac{\cosh\left(|k_\perp| \cdot |\vec{x}-\vec{x}'|\right)}{|k_\perp|} e^{-k_\perp^2 |t-t'|}$$

$$\sim \frac{1}{|\vec{x}-\vec{x}'|^{d-2}} \int_0^\infty dr \cdot r^{d-3} \cosh(r) e^{-\alpha r^2} \text{ with } \alpha = \frac{2|t-t'|}{(\vec{x}-\vec{x}')^2}$$

$$d=1 \rightarrow \int_0^\infty dr \, \frac{\cosh(r) e^{-\alpha r^2}}{r^2}$$ (diverges)

$$d=2 \rightarrow \int_0^\infty dr \, \frac{\cosh(r) e^{-\alpha r^2}}{r}$$ (logarithmically diverges)

$$d=3 \rightarrow \int_0^\infty dr \cosh(r) e^{-\alpha r^2} \cdot |\vec{x}-\vec{x}'|$$

when $|x - x'| \to 0$, $\alpha \to \infty \Rightarrow \cosh(r) \sim 1$

and $\int_0^\infty dr \, e^{-\alpha r^2} \sim \frac{1}{\sqrt{\alpha}}$,

$$\langle h(\vec{x},t) \cdot h(\vec{x}',t') \rangle \sim \frac{|\vec{x}-\vec{x}'|^2}{|t-t'|}$$

$\therefore$ If random process then surface $h(\vec{x},t)$ would not show correlations (surface is smooth in long range limit) for $d = 1$ or 2 however, surface is rough since divergences indicate correlations.