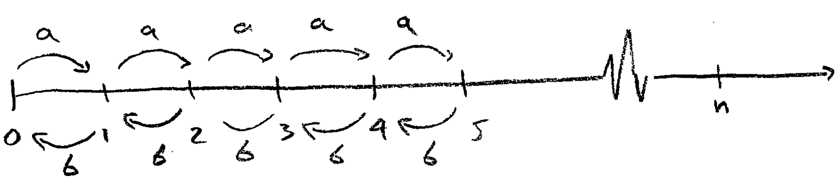
(1)



[boundary condition] $\dot{p}_0 = b p_1 - a p_0$ [initial condition]: $P_n(t) = \delta_{n0}$

$$\dot{p}_n = a p_{n-1} + b p_{n+1} - (a+b) p_n \quad (\text{for } n>0)$$

Let $a+b = 1$ for simplicity (similiar to setting $\omega = 1$ w/o loss of generality in previous problems)

$a = \sin^2\theta, \quad b = \cos^2\theta \qquad \theta < \frac{\pi}{4}$

Laplace Transform:

(1) $s p_0 - 1 = b p_1 - a p_0$

(2) $s p_n = a p_{n-1} + b p_{n+1} - p_n$ (for $n>0$)

Guess solution form: $p_n = \alpha^n$, solve for α in (2):

$$b\alpha^2 - (s+1)\alpha + a = 0 \qquad \alpha_\pm = \frac{(s+1) \pm \sqrt{(s+1)^2 - 4ab}}{2b}$$

General solution $\quad p_n(s) = A(\alpha_+)^n + B(\alpha_-)^n$

Apply Initial condition:

$$P_n(t) = \delta_{n_0} \rightarrow \lim_{s\to\infty} P_n(s) = \lim_{s\to\infty} \frac{\delta_{n0}}{s} = 0$$

$$\lim_{s\to\infty} (\alpha_-)^n = \left(\frac{s-s}{2b}\right)^n = 0 \qquad \lim_{s\to\infty} (\alpha_+)^n = \left(\frac{s+s}{2b}\right)^n = \infty$$

diverges at $t=0$

Thus $\quad P_n(s) = B(\alpha_-)^n$ (for $n>0$)

Match the solution with the boundary condition.

$$P_0(s) = B, \quad sB - 1 = bB\alpha_- - aB \rightarrow B = \frac{1}{(s+a) - b\alpha_-}$$

$$P_n(s) = \frac{(\alpha_-)^n}{(s+a) - b\alpha_-}$$

① continued.

Long time limit $t \to \infty \iff s \to 0$

$$\lim_{s\to 0}\left((s+1)^2-4ab\right)^{\frac{1}{2}}$$ use $a+b=1$ to get into better form:

$$= \lim_{s\to 0}\left((s+a+b)^2-4ab\right)^{\frac{1}{2}} = \lim_{s\to 0}\left(\cancel{s^2}+2(a+b)s+(a+b)^2-4ab\right)^{\frac{1}{2}}$$

drops off faster than s

$$= \lim_{s\to 0}\left(2(a+b)s+(b-a)^2\right)^{\frac{1}{2}}$$

now, complete the square with an addition of a desirable s^2 term:

$$= \lim_{s\to 0}\left(2(a+b)s+(b-a)^2+\left(\frac{a+b}{b-a}\right)^2 s^2\right)^{\frac{1}{2}} = \lim_{s\to 0}\left[(b-a)+\left(\frac{\cancel{a+b}\;1}{b-a}\right)s\right]$$

$$= \lim_{s\to 0}\left[(b-a)+\frac{s}{b-a}\right]$$

$$\lim_{s\to 0}\alpha_- = \frac{s+1-\left(b-a+\frac{s}{b-a}\right)}{2b} = \frac{a}{b}\left(1-\frac{s}{b-a}\right)$$

$$\lim_{s\to 0}P_0(s) = \frac{1}{s+a-b\frac{a}{b}\left(1-\frac{s}{b-a}\right)} = \frac{b-a}{bs}$$ diverges!

thus, $P_n(s) = \dfrac{(\alpha_-)^n}{(s+a)-b\alpha_-}$ diverges for $t\to\infty$ and is valid for $t\to 0$

Conversly, $P_n(s) = \dfrac{(\alpha_+)^n}{s+a-b\alpha_+}$ converges for $t\to\infty$ and diverges for $t\to 0$

$$\lim_{s\to 0}\alpha_+ = \frac{s+1+\left(b-a+\frac{s}{b-a}\right)}{2b} = 1+\frac{s}{b-a}$$

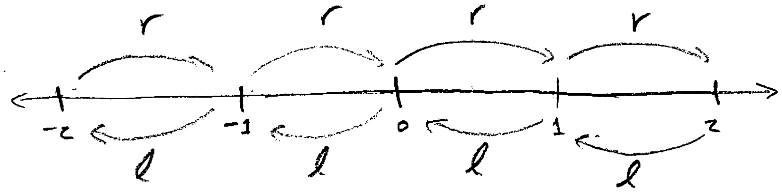
$$\lim_{s\to 0}P_0(s) = \frac{1}{s+a+b\left(1+\frac{s}{b-a}\right)} = \frac{(b-a)}{(b-a)+s(2b-a)} \Rightarrow P_0(t)\sim e^{-\left(\frac{b-a}{2b-a}\right)t}$$

$$\lim_{s\to 0}P_n(s) \quad \frac{\left(1+\frac{s}{b-a}\right)^n}{s+a+b\left(1+\frac{s}{b-a}\right)} = \frac{n}{2b-a}+O(s) \Rightarrow P_{n>0}(t)\sim\left(\frac{n}{2b-a}\right)\frac{1}{t}$$

for $n>0$

2a.

Master Equation.



$$\dot{p}_n = l p_{n+1} + r p_{n-1} - (r+l) p_n$$

Fourier Transform $\quad p(k,t) = \sum_{-\infty}^{\infty} p_n(t) e^{ikn}$

$$\dot{p}(k,t) = \left[l e^{-ik} + r e^{ik} - (r+l) \right] p(k,t)$$

Let $\quad s = l + r$, $\quad d = l - r$ $\quad \Big\} \quad l = (s+d)/2$, $\quad r = (s-d)/2$

$$\dot{p}(k,t) = \left[\left(\frac{s+d}{2}\right) e^{-ik} + \left(\frac{s-d}{2}\right) e^{ik} - s \right] p(k,t)$$

$$= \left[s(\cos k - 1) - \frac{id}{2} \sin k \right] p(k,t)$$

Let $\quad r = 1 + \varepsilon$, $\quad l = 1 - \varepsilon$ $\quad \Big\} \quad s = 2$, $\quad d = -2\varepsilon$

Initial condition $\quad p(k, t=0) = 1$

$$p(k,t) = e^{[2(\cos k - 1) + i\varepsilon \sin k]t}$$

using $\quad e^{2t\cos k} = \sum_{-\infty}^{\infty} e^{ikn} I_n(2t)$ and.

$$e^{t(i\varepsilon \sin k)} = I_0(t) + \sum_{n=0}^{\infty} (-1)^n I_{2n+1}(t) \left[e^{(2n+1)ki} - e^{-(2n+1)ki} \right] + \sum_{n=1}^{\infty} (-1)^n I_{2n}(t) \left[e^{2nki} + e^{-2nki} \right]$$

from Abramowitz + Slegun pg 376

$\therefore \; p(k,t) =$

$$e^{-2t} \sum_{-\infty}^{\infty} e^{ikn} I_n(2t) \Big[\quad \downarrow \quad \Big]$$

multiply out series to determine coeffiecients of e^{ikn} to deduce $p_n(t)$

② b

$$\langle n^\theta \rangle = \sum_{-\infty}^{\infty} n^\theta p_n(t)$$

$$= \sum_{-\infty}^{\infty} n^\theta e^{ikn} p_n(t) \Big|_{k=0}$$

$$= \frac{\partial^\theta}{\partial^\theta (ik)} \sum e^{ikn} p_n(t) \Big|_{k=0} = \frac{\partial^\theta}{\partial^\theta (ik)} p(k,t) \Big|_{k=0}$$

$$\langle n \rangle = \frac{\partial}{\partial (ik)} e^{[Z(\cos k - 1) + i\varepsilon \sin k]t} \Big|_{k=0}$$

$$= -i\, e^{[Z(\cos k - 1) + i\varepsilon \sin k]t} \left(-Z\sin k + i\varepsilon \cos k\right) t \Big|_{k=0}$$

$$= \varepsilon t$$

$$\langle n^2 \rangle = -\frac{\partial^2}{\partial k^2} e^{[Z(\cos k - 1) + i\varepsilon \sin k]t} \Big|_{k=0}$$

$$= -e^{[Z(\cos k - 1) + i\varepsilon \sin k]t} \Big[\left(\left(-Z\sin k + i\varepsilon\cos k\right) t \right)^2 + \left(-Z\cos k - i\varepsilon \sin k\right) t \Big] \Big|_k$$

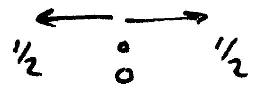
$$= +\varepsilon^2 t^2 + Zt$$

Note: $\langle n^2 \rangle - \langle n \rangle^2 = Zt$

See : "Random Walk with Shrinking Steps", Krapivsky and Redner on course web site.

3. a.

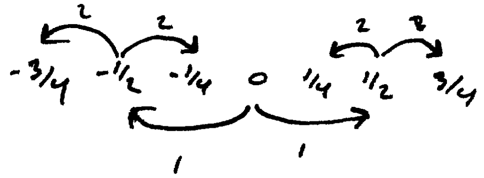
$n=1, \quad \lambda^1 = \frac{1}{2}$



$P(\frac{1}{2}) = \frac{1}{2}$

$P(-\frac{1}{2}) = \frac{1}{2}$

$n=2, \quad \lambda^2 = \frac{1}{4}$



$$P(x) = \begin{cases} \frac{1}{4}, & x = -3/4, -1/4, 1/4, 3/4 \\ 0 & \text{otherwise} \end{cases}$$

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1. $P(x;N) = \sum\limits_{x'} P(x-x'; N-1)\, p_N(x')$

- in order to reach x at step N, the walker must reach $x-x'$ at step $N-1$, then hop a distance x'.

2. $p_n(k) = \int_{-\infty}^{\infty} p_n(x)\, e^{ikx}\, dx$

$P(k;N) = \int_{-\infty}^{\infty} P(x;N)\, e^{ikx}\, dx$

Sub. 2 into 1,

$$\Rightarrow \quad P(k;N) = P(k;N-1)\, p_N(k)$$

$$= P(k;0) \prod_{n=0}^{\infty} p_n(k)$$

Since we start at the origin, $P(x;0) = \delta_{x,0}$

so $P(k;0) = 1$

4.

$$\frac{dS(x)}{dt} = -S(x) + \frac{1}{z} \sum_i S(\vec{x} + \vec{e}_i)$$

$z = 2d$ for d-dimensional lattice

e_i are the unit vectors on the lattice

In 1-D,

$$\dot{P}_n = \frac{\gamma}{2}(P_{n-1} + P_{n+1}) - P_n$$

has sol'n.

$$P_n(t) = I_n(\gamma t)e^{-t}$$

through Fourier transform (see ch. 7 of Fundamental kinetic Processes, eq. 7.6 – 7.10)

In d-dimension, the Fourier transform factorizes similarly

to give $S(x,t) = \prod_{i=1}^{d} I_{x_i}(t)\, e^{-dt}$

$$\sim \frac{1}{(2\pi t)^{d/2}}$$

So the opinion of the voter at the origin relaxes to the average of the rest of the population — undecided.