542 HW 2

1. a) The Boltzmann transport eq'n. (BTE) is

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_r + \vec{F} \cdot \vec{\nabla}_p \right) f(\vec{r}, \vec{p}, \vec{t}) = 0$$

If we assume a local Maxwell-Boltzmann (MB) distribution, we have the ansatz

$$f(r,p,t) = \underbrace{f_{eq}(p)}_{\substack{MB \\ dist.}} \underbrace{e^{-\phi(r)/kT}}_{\substack{Boltzmann \\ factor}} , \quad \phi = mgz$$

In steady state, $\frac{\partial f}{\partial t} = 0$,

so $\left(\vec{v} \cdot \vec{\nabla}_r + \vec{F} \cdot \vec{\nabla}_p \right) f = 0$

$$\xrightarrow{1.} \left(\vec{v} \cdot \vec{\nabla}_r \right) f = f_{eq}\, e^{-\phi/kT} \left(\frac{-\vec{v} \cdot \vec{\nabla} \phi}{kT} \right)$$

$$= f_{eq}\, e^{-\phi/kT} \left(\frac{+\vec{v} \cdot \vec{F}}{kT} \right) \qquad \left(\vec{F} = -\vec{\nabla} \phi \right)$$

$$\xrightarrow{2.} \left(\vec{F} \cdot \vec{\nabla}_p \right) \left(f_{eq}\, e^{-\phi/kT} \right) = f_{eq}\, e^{-\phi/kT} \left(\frac{-\vec{v} \cdot \vec{F}}{kT} \right)$$

1 & 2 cancel, so $f(r,p,t)$ solves BTE,

$$n(z) = n_0\, e^{-\lambda z}$$

w/ $n_0 = f_{eq}(p)$, $\lambda = \frac{mg}{kT}$

2a

$$T(z,t) = \int_{-\infty}^{\infty} S(z-y,t)\,\phi(y)\,dy \quad \text{— convolution} \qquad , \; S(x) = \frac{1}{2\sqrt{\pi\kappa t}}\, e^{-\frac{x^2}{4\kappa t}} \quad \text{— Green's fn.}$$

$$\phi(y) = \text{I.C.} = T(y,0)$$

for $T(0,t) = 0$, $T(z,t)$ must be odd:

$$T(z,0) = \begin{cases} T_0 & z<0 \\ -T_0 & z>0 \\ 0 & z=0 \end{cases} \quad \text{— B.C.}$$

$$T(z,t) = \int_{-\infty}^{0} S(z-y,t)\, T_0\, dy + \int_{0}^{\infty} S(z-y,t)(-T_0)\, dy$$

$$= \frac{T_0}{2\sqrt{\pi\kappa t}}\left[\int_{-\infty}^{0} e^{-\frac{(z-y)^2}{4\kappa t}}\, dy - \int_{0}^{\infty} e^{-(z-y)^2/4\kappa t}\, dy\right]$$

$$= \frac{T_0}{2\sqrt{\pi\kappa t}}\left[\int_{-\infty}^{0} e^{-\frac{(z-y)^2}{4\kappa t}}\, dy - \int_{0}^{-\infty} e^{-(z+y)^2/4\kappa t}\, (-dy)\right]$$

$$(y \to -y)$$

$$= \frac{T_0}{2\sqrt{\pi\kappa t}}\left[\int_{-\infty}^{0} e^{-\frac{(z-y)^2}{4\kappa t}}\, dy - \int_{-\infty}^{0} e^{-(z+y)^2/4\kappa t}\, dy\right]$$

$$T(z,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{0} \left(e^{-(z-z')^2/4\kappa t} - e^{-(z+z')^2/4\kappa t}\right) dz'$$

Check

✓ B.C. : $T(0,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int e^{-z'^2/4\kappa t} - e^{-z'^2/4\kappa t} \;\underline{= 0}$ ✓

I.C. : $T(z,t)\Big|_{t\to 0} = \frac{T_0}{2\sqrt{\pi\kappa t}} \cdot \left[\frac{\sqrt{4\pi\kappa t}}{2} + \frac{\sqrt{4\pi\kappa t}}{2}\right] = \underline{T_0}$ ✓

2b)

$$\vec{F}_\infty = -(6\pi\eta R)\vec{v}_\infty + mg = 0$$

$$\Rightarrow v_\infty = \frac{mg}{6\pi\eta R}$$

c)

$$\vec{J} = D\frac{\partial f}{\partial z}$$

$$= D\frac{mg}{kT} f_{eq}(p)\, e^{-mgz/kT}$$

d) At equilibrium, $J = f v_\infty + D\frac{\partial f}{\partial z} = 0$

equating, $D = \frac{kT}{6\pi\eta R}$

2 b) $$\left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{T_0}{2\sqrt{\pi \kappa t}} \int -2(z-z')\, e^{-(z-z')^2/4\kappa t} - -2(z+z')\, e^{-(z+z')^2/4\kappa t}\, dz'$$

$$= \frac{T_0}{2\sqrt{\pi \kappa t}} \cdot 2 \int_{-\infty}^{0} z' e^{-z'^2/4\kappa t}\, dz'$$

$$= -\frac{T_0}{\sqrt{\pi \kappa t}} \sim \frac{10^3}{\sqrt{10^6 \cdot 10^{16}}}$$

$\sim$ age of Earth $\sim 10^9$ yrs. $\times 10^7$ sec.

$$\sim 10^{-8}\ {}^\circ C/m$$

- But we've ignored all energy sources:

the Sun and the Earth's core.

2c

$$T(\vec{r},t) = \int_{\mathbb{R}^3} S(\vec{r}-\vec{r}',t)\,\phi(\vec{r}')\,d\vec{r}' \quad \text{— convolution}$$

$$S(\vec{r},\vec{r}') = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-(\vec{r}-\vec{r}')^2/4\kappa t} \quad \text{— Green's fn.}$$

$$\phi(\vec{r}') = \begin{cases} T_0 & r < R \\ 0 & r = R \\ -T_0 & r > R \end{cases} \quad \text{— B.C.}$$

so $$T(\vec{r},t) = \int_{|\vec{r}'|<R} \frac{1}{(4\pi\kappa t)^{3/2}} e^{-(\vec{r}-\vec{r}')^2/4\kappa t} \cdot T_0 \, d\vec{r}'$$

$$+ \int_{|\vec{r}'|>R} \frac{1}{(4\pi\kappa t)^{3/2}} e^{-(\vec{r}-\vec{r}')^2/4\kappa t} (-T_0)\, d\vec{r}'$$

$$= \frac{T_0}{(4\pi\kappa t)^{3/2}} \left[\int_\theta \int_\phi \int_{r'<R} e^{-(\vec{r}-\vec{r}')^2/4\kappa t} d\vec{r}' \right.$$

$$\left. - \int_\theta \int_\phi \int_{r'>R} e^{-(\vec{r}-\vec{r}')/4\kappa t} d\vec{r}' \right]$$

choose $\vec{r}$ along z-axis

so

$$T(\vec{r},t) = \frac{2\pi T_0}{(4\pi\kappa t)^{3/2}} \int_0^\pi \sin\theta\, d\theta \left[\int_0^R e^{-(r^2+r'^2-2rr'\cos\theta)} \cdot r'^2 dr' \right.$$

$$\left. - \int_R^\infty e^{-(r^2+r'^2-2rr'\cos\theta)} r'^2 dr' \right]$$