[redacted]

9.3. (a) Using the thermodynamic relation

$$C_P - C_V = T(\partial P/\partial T)_V(\partial V/\partial T)_P = -T(\partial P/\partial T)_V^2/(\partial P/\partial V)_T$$

and the equation of state (9.3.9), we get

$$\frac{C_P - C_V}{Nk} = -\frac{T(\partial P/\partial T)_\mathrm{v}^2}{k(\partial P/\partial \mathrm{v})_T} = -\frac{T\{k/(\mathrm{v}-b)\}^2}{k\{-kT/(\mathrm{v}-b)^2 + 2a/\mathrm{v}^3\}} = \frac{1}{1-2a(\mathrm{v}-b)^2/kT\mathrm{v}^3} .$$

(b) In view of the thermodynamic relation

$$TdS = C_V\, dT + T(\partial P/\partial T)_V dV$$

and the equation of state (9.3.9), an adiabatic process is characterized by the fact that

$$C_V dT + NkT(\mathrm{v}-b)^{-1} d\mathrm{v} = 0 .$$

Integrating this result, under the assumption that $C_V = const.$, we get

$$T^{C_V/Nk}(\mathrm{v}-b) = const.$$

(c) For this process we evaluate the Joule coefficient

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{(\partial U/\partial V)_T}{(\partial U/\partial T)_V} = -\frac{T(\partial P/\partial T)_V - P}{C_V} = -\frac{a/\mathrm{v}^2}{C_V} = -\frac{N^2 a}{C_V V^2} .$$

Now integrating from state 1 to state 2, we readily obtain the desired result.

[redacted]

2.

$$P/P_{sea} = \exp\left(-\frac{Mg}{RT} z\right) \approx \exp\left(-\frac{1}{2}\right)$$

$$\frac{dP}{dT} = P\frac{L}{kT^2} \Rightarrow \int \frac{dP}{P} = \int \frac{L}{kT^2}\, dT$$

$$\ln P - \ln P_{sea} = -\frac{L}{kT} + \frac{L}{kT_{sea}}$$

$$\Rightarrow T = \left[\frac{1}{T_{sea}} \cancel{+} - \frac{k}{L}\ln\left(\frac{P}{P_{sea}}\right)\right]^{-1}$$

$$T_{sea} = 373K \qquad \Rightarrow T \approx 359K = 86°C$$

3. (a) let $\tau_i = \sigma_i \sigma_{i+1} = \pm 1$

$$\mathcal{H} = -J_1 \sum \sigma_i \sigma_{i+1} - J_2 \sum \sigma_i \sigma_{i+1}^2 \sigma_{i+2}$$
$$= -J_1 \sum \tau_i - J_2 \sum \tau_i \tau_{i+1} \quad (\because \sigma_i^2 = 1)$$

$$Z_N = \sum e^{-\beta H} = Tr(T^N) = \lambda_-^N + \lambda_+^N.$$

$$T = \begin{pmatrix} ++ , & +- \\ -+ , & -- \end{pmatrix}$$

$\because \lambda_\pm$ are eigenvalues of T

$$0 = \begin{vmatrix} e^{\beta(J_2+J_1)} - \lambda & e^{-\beta J_2} \\ e^{-\beta J_2} & e^{\beta(J_2 - J_1)} - \lambda \end{vmatrix}$$

$$= \lambda^2 - 2\lambda e^{\beta J_2} \cosh(\beta J_1) + 2\sinh(2\beta J_2)$$

$$\Rightarrow \lambda_\pm = e^{\beta J_2} \cosh(\beta J_1) \pm \sqrt{e^{2\beta J_2} \cosh^2(\beta J_1) - 2\sinh(2\beta J_2)}$$

$\because \lambda_+ > \lambda_- \quad N \to \infty \quad \lambda_+^N \gg \lambda_-^N$

$$Z_N \cong \lambda_+^N = \left\{ e^{\beta J_2} \cosh(\beta J_1) + \sqrt{e^{2\beta J_2} \cosh^2(\beta J_1) - 2\sinh(2\beta J_2)} \right\}$$

(b) for $T=0$. $\beta \to \infty$

$$Z_N = \left(\frac{1}{2} e^{\beta J_2} e^{\beta J_1} + \sqrt{\frac{1}{4}(e^{\beta J_2})^2 (e^{\beta J_1})^2 - e^{2\beta J_2} + e^{-2\beta J_2}}\right)$$

for $J_2 > -J_1$

$$Z_N \simeq e^{N\beta(J_1+J_2)} \Rightarrow U = -N(J_1+J_2)$$

for $J_2 \leq -J_1$ $\quad Z_N \simeq e^{-N\beta J_2} \Rightarrow U = NJ_2$

for $J_2 > -J_1$ case $\quad U = -N(J_1+J_2) = \langle -J_1 \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - J_2 \sum \sigma_i \sigma_{i+2} \rangle$

$\Rightarrow \sigma_i \sigma_{i+1} = 1$ and $\sigma_i \sigma_{i+2} = 1$ $\Rightarrow$ $\uparrow\uparrow\uparrow\uparrow$ state

ferromagnetic behavior

for $J_2 \leq -J_1$ we can get $\sigma_i \sigma_{i+1} = \pm 1$ $\Rightarrow$ $\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow$ state length l of a polymer with the end-to-end distance of a random walk in the following way

$$l = \sqrt{\langle (x_N - x_0)^2 \rangle} = \sqrt{N + \sum_{i \neq j} \sigma_i \sigma_j} .$$

In the diagram we see that the length increases as function of temperature and we can therefore assume that at very low temperatures, neighboring spins prefer to point in opposite directions so as to shorten the walk. Therefore a possible model is the Ising antiferromagnet with

$$H = J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} .$$

The correlation function for this model can be worked out in the same fashion as for the ferromagnet with the result

$$\langle \sigma_i \sigma_j \rangle = (-v)^{|j-i|}$$

where $v = \tanh K$ with $K = \beta J$. Therefore

$$l^2 = N + 2 \sum_{i=2}^{N} \sum_{j=1}^{i-1} (-v)^{i-j} .$$

Retaining only terms proportional to N we arrive at the result

$$l = \sqrt{N \frac{1-v}{1+v}} = \sqrt{N} e^{-K}$$

which shows the expected increase of the length of the chain as function of T.

3.7

The elements of the transfer matrix for the spin-1 Ising ferromagnet are given by

$$P_{\sigma,\sigma'} = \exp K\sigma\sigma'$$

where $K = \beta J$ and $\sigma = 0, \pm 1$. Therefore

$$\mathbf{P} = \begin{bmatrix} e^K & 1 & e^{-K} \\ 1 & 1 & 1 \\ e^{-K} & 1 & e^K \end{bmatrix} .$$

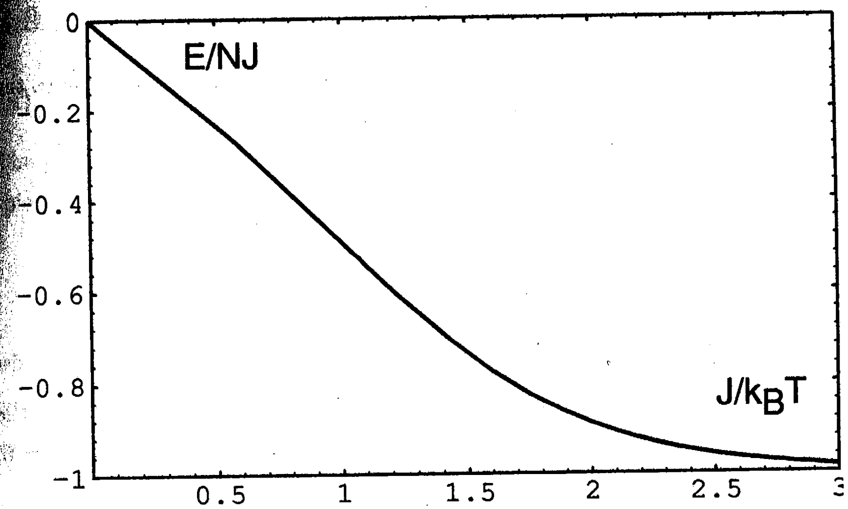

Figure 3.3: Plot of internal energy vs. inverse temperature in Problem 3.7.

The eigenvalues of this matrix can be obtained (e.g. using "Mathematica" as

$$\lambda_1 = \frac{x^2 - 1}{x}$$

$$\lambda_2 = \frac{1}{2x}\left(1 + x + x^2 + \sqrt{1 - 2x + 11x^2 - 2x^3 + x^4}\right)$$

$$\lambda_3 = \frac{1}{2x}\left(1 + x + x^2 - \sqrt{1 - 2x + 11x^2 - 2x^3 + x^4}\right)$$

where $x = e^K$. The largest eigenvalue is λ_2. The internal energy is thus

$$E = -NJx \frac{\partial \ln \lambda_2}{\partial x}$$

$$= NJ \left(1 - \frac{x + 2x^2 + \frac{-2x+22x^2-6x^3+4x^4}{2\sqrt{1-2x+11x^2-2x^3+x^4}}}{1 + x + x^2 + \sqrt{1 - 2x + 11x^2 - 2x^3 + x^4}} \right) .$$

A plot of the internal energy vs the inverse temperature is shown in Figure 3.3.