9.16

$\bar{v}_x = 0$ by symmetry.

By definition,

$$\overline{v_x^2} = \frac{\int d^3\underline{v}\, f(\underline{v})\, v_x^2}{\int d^3\underline{v}\, f(\underline{v})} \qquad (1)$$

At absolute zero, the particles fill all the lowest states so that the distribution $f(\underline{v})$ is just a constant and cancels in (1). The maximum speed is at $\frac{1}{2} m v_F^2 = \mu$. By symmetry, $\overline{v_x^2} = \frac{1}{3}\overline{v^2}$ and (1) becomes

$$\overline{v_x^2} = \frac{\frac{4\pi}{3}\int_0^{v_F} v^4\, dv}{4\pi \int_0^{v_F} v^2\, dv} = \frac{1}{5} v_F^2 = \frac{2}{5}\frac{\mu}{m}$$

**8.7.** This problem is similar to Problem 7.9 of the Bose gas and can be done the same way. At low temperatures, using formula (E.15), we get

$$\langle u\rangle\langle u^{-1}\rangle = \frac{9}{8}\left\{1+\frac{\pi^2}{3}(\ell n\, z)^{-2}+\ldots\right\}\left\{1+\frac{\pi^2}{8}(\ell n\, z)^{-2}+\ldots\right\}^{-2}$$



$$= \frac{3}{2n(kT\,\ell n\, z)}\left\{1-\frac{\pi^2}{6}(\ell n\, z)^{-2}+\ldots\right\}.$$

We now employ eqn. (8.1.35) and get

**8.14.** In the notation of Sec. 3.9, the potential energy of a magnetic dipole in the presence of a magnetic field $\boldsymbol{B} = (0,0,B)$ is given by the expression $-(g\mu_B m)B$, where $m = -J,\ldots,+J$. The total energy $\varepsilon$ of the dipole is then given by $\varepsilon = (p^2/2m') - g\mu_B mB$, $m'$ being the (effective) mass of the particle; the momentum of the particle may then be written as

$$p = \{2m'(\varepsilon + g\mu_B mB)\}^{1/2}.$$

At $T = 0$, the number of such particles in the gas will be



$$N_m = \frac{4\pi V}{3h^3}\{2m'(\varepsilon_F + g\mu_B mB)\}^{3/2}$$

and hence the net magnetic moment of the gas will be given by

$$M = \sum_m (g\mu_B m)N_m = \frac{4\pi g\mu_B V}{3h^3}(2m')^{3/2}\sum_m m(\varepsilon_F + g\mu_B mB)^{3/2}.$$

We thus obtain for the *low-field* susceptibility (per unit volume) of the system

$$\chi_0 = \underset{B\to 0}{Lim}\left(\frac{M}{VB}\right) = \frac{4\pi g\mu_B}{3h^3}(2m')^{3/2}\cdot\frac{3}{2}g\mu_B\varepsilon_F^{1/2}\sum_{m=-J}^{J} m^2$$

$$= \frac{2\pi g^2\mu_B^2}{3h^3}(2m')^{3/2}\varepsilon_F^{1/2}J(J+1)(2J+1). \qquad (1)$$

By eqn. (8.1.24),

$$\varepsilon_F^{3/2} = \frac{3n}{4\pi(2J+1)}\frac{h^3}{(2m')^{3/2}} \qquad \left(n = \frac{N}{V}\right). \qquad (2)$$

Substituting (2) into (1), we obtain the desired result

$$\chi_0 = \frac{1}{2}n\mu^{*2}/\varepsilon_F \qquad \{\mu^{*2} = g^2\mu_B^2 J(J+1)\}.$$

With $g = 2$ and $J = 1/2$, we obtain: $\chi_0 = (3/2)n\mu_B^2/\varepsilon_F$, in agreement with eqn. (8.2.6).

The corresponding result in the limit $T \to \infty$ is given by

$$\chi_\infty = \frac{1}{3}n\mu^{*2}/kT;$$

see eqn. (3.9.26). We note that the ratio $\chi_0/\chi_\infty = 3kT/2\varepsilon_F$, valid for all $J$.

**8.19.** Utilizing the result obtained in Problem 8.13, we have for a Fermi gas at low temperatures

$$\frac{C_V}{Nk} = \frac{\pi^2}{3}\frac{a(\varepsilon_F)}{N}kT \,. \qquad (1)$$

Now, the density of states for the relativistic gas is given by, see eqn. (8.4.7),

$$a(\varepsilon) = \frac{8\pi V}{h^3}p^2\frac{dp}{d\varepsilon} = \frac{8\pi m V}{h^3}p\left\{1+\left(\frac{p}{mc}\right)^2\right\}^{1/2},$$

where $p = p(\varepsilon)$. Substituting this result into (1) and making use of eqn. (8.4.4), we get

$$\frac{C_V}{Nk} = \frac{\pi^2 m}{p_F^2}\left\{1+\left(\frac{p_F}{mc}\right)^2\right\}^{1/2} kT \,,$$

which leads to the desired result.

In the non-relativistic case ($p_F \ll mc$ and $\varepsilon_F = p_F^2/2m$), we obtain the familiar expression (8.1.39); in the extreme relativistic case ($p_F \gg mc$ and $\varepsilon = pc$), we obtain

$$\frac{C_V}{Nk} = \pi^2\left(\frac{kT}{\varepsilon_F}\right),$$

consistent with expression (7) of the solution to Problem 8.13.